Quantum electrodynamics on a space–time lattice

Jerzy Kijowski * , Artur Thielmann

Center for Theoretical Physics, Polish Academy of Sciences,
Al. Lotników 32/46, PL-02-668 Warsaw, Poland

Abstract

A Minkowski-lattice version of quantum electrodynamics (or rather its simplified version, with matter described by a scalar field) is constructed. Quantum fields are consequently described in a gauge-independent way, i.e. the algebra of quantum observables of the theory is generated by gauge-invariant operators assigned to zero-, one-, and two-dimensional elements of the lattice. The operators satisfy canonical commutation relations. The uniqueness of representation of this algebra is proved. Field dynamics is formulated in terms of difference equations imposed on the field operators. It is obtained from a discrete version of the path-integral. The theory is local and causal.

0. Introduction

Standard difficulties of quantum field theory (renormalization, regularization of the product of fields at the same space–time point, etc.) are augmented in the case of gauge fields by the gauge-fixing problem. Having at our disposal no satisfactory non-perturbative description, one commonly uses non-physical objects like ghosts, bare particles, etc.

In the present paper we show how to solve completely the gauge-invariance problem, at least at the level of lattice approximation, for the theory of interacting scalar charged field ϕ and the electromagnetic field, described by the vector-potential A_μ. We use the real-time lattice, not its euclidean version. This way we maintain the essential structure of the Quantum Field Theory: the local character of quantum field operators fulfilling the

---

* Corresponding author.

canonical commutation relations, local time evolution given by a unitary transformation, the causal (light-cone) character of field dynamics. As a result we obtain a discrete version of an algebraic Quantum Field Theory, where, similarly as in Haag formulation (see [7]), the quantum field is described in terms of algebras of local observables, assigned to bounded regions of the lattice. The quantum dynamics is formulated in terms of difference equations, which have to be satisfied by the field operators. The equations have causal character and the initial value problem is well posed. In particular, operators describing electric and magnetic field fulfill the Maxwell equations with the continuous operators **div** and **curl** replaced by their obvious lattice analogs.

Of course, the problem "how to remove the cut off introduced by the finite lattice spacing and to construct an exact, continuous model of interacting gauge-fields?" is several orders of magnitude more difficult. We do not discuss it here. But we believe that, due to its simple structure, our model may be very useful for understanding the fundamental structures of the quantum gauge field theory.

Our construction is based on the following, heuristic point of view, which we found extremely useful as a guiding principle. Consider two different Cauchy surfaces: $\Sigma_{\text{init}} = \{x^0 = t_{\text{init}}\}$ and $\Sigma_{\text{fin}} = \{x^0 = t_{\text{fin}}\}$ in the Minkowski space–time M. The classical dynamics gives the relation between the initial data on Σ_{init} and the final data on Σ_{fin}. We want to define its quantum counterpart by the path integral

$$(\Psi_{\text{fin}}|\Psi_{\text{init}}) = \frac{1}{N} \int \overline{\Psi}_{\text{fin}} \exp\left(\frac{i}{\hbar} \int \mathcal{L} \, d^4 y\right) \Psi_{\text{init}} \prod_{x_{\text{fin}}} dA(x_{\text{fin}}) \, d\phi(x_{\text{fin}}) \, d\bar{\phi}(x_{\text{fin}})$$
$$\times \prod_{y} dA(y) \, d\phi(y) \, d\bar{\phi}(y) \prod_{x_{\text{init}}} dA(x_{\text{init}}) \, d\phi(x_{\text{init}}) \, d\bar{\phi}(x_{\text{init}}) . \tag{0.1}$$

Here, the initial (resp. the final) quantum state Ψ is represented by a "wave function" Ψ depending on initial configurations $A(x_{\text{init}})$ and $\phi(x_{\text{init}})$ (resp. final configurations $A(x_{\text{fin}})$ and $\phi(x_{\text{fin}})$). The functional integral may be thought naively as being performed over three groups of variables: initial configurations $(A(x_{\text{init}}), \phi(x_{\text{init}}))$, intermediate configurations $(A(y), \phi(y))$ and the final configurations $(A(x_{\text{fin}}), \phi(x_{\text{fin}}))$. Performing the integration over the first and the second group, with the last one being fixed, should give us the unitary evolution kernel $U_{(\Sigma_{\text{fin}}, \Sigma_{\text{init}})}$ from Σ_{init} to Σ_{fin}:

$$U_{(\Sigma_{\text{fin}}, \Sigma_{\text{init}})} \Psi_{\text{init}} = \frac{1}{N} \int \exp\left(\frac{i}{\hbar} \int \mathcal{L} \, d^4 y\right) \Psi_{\text{init}} \prod_{y} dA(y) \, d\phi(y) \, d\bar{\phi}(y)$$
$$\times \prod_{x_{\text{init}}} dA(x_{\text{init}}) \, d\phi(x_{\text{init}}) \, d\bar{\phi}(x_{\text{init}}). \tag{0.2}$$

The goal of the present paper is to construct a rigorous, discrete version of the above functional integral. For this purpose let us imagine that the region of space–time contained between Σ_{init} and Σ_{fin} is divided into small four-dimensional hypercubes. This discretization induces also a partition of both Σ_{init} and Σ_{fin} into three-dimensional cubes. In a discretized version of the theory the quantum state on any Σ will be defined as a wave function depending on a discrete number of degrees of freedom related to those three-dimensional cubes. We

define the physical quantum states as those, which fulfill the discrete version of constraint equations, relating the divergence of the electric induction field to the electric charge carried by the matter field (see [4] and the discussion therein). At the level of the continuous theory this constraint is described by Eq. (1.9) and by Eq. (3.12) in the discrete version. This requirement implies that the physically admissible wave functions have to be gauge-invariant on Σ' s. Hence, they can be considered as functions of a complete set of gauge-invariants. The first step of our construction consists in organizing such gauge-invariant functions into a Hilbert space. This way we obtain a lattice version of gauge-invariant quantum kinematics.

To define the dynamics of such a theory we have to pass through the following three steps:

(1) We discretize the action on the lattice. Since the action does not change under gauge transformations, it can also be expressed in terms of a complete set of gauge-invariant observables.

(2) We have to remove from (0.2) an infinite factor, resulting from the integration over the gauge parameters, and to leave only the integration over "true degrees of freedom". This has to be done not only for intermediate configurations $(A(y), \phi(y))$ but also for the initial configurations. This way the initial arguments of the evolution kernel defined by (0.2) match the (gauge-invariant) arguments of the initial wave function representing the quantum state of the system.

(3) Since both the initial wave function and the Lagrangian are gauge-invariant, we obtain from (0.2) a gauge-invariant final wave function representing the physical quantum state.

The integral we use in the second step of this procedure is, in fact, defined on the "gauge-orbit space", i.e. over the quotient of the configuration space modulo the action of the gauge group. There have been many proposals to define a natural measure on this space. In the present paper we use the measure proposed in [19]. But we prove in Section 4.3 a theorem about the *uniqueness* (up to equivalence) of the irreducible representation of the observable algebra obtained this way. This result fully justifies, in our opinion, the correctness of our approach.

It turns out that the dynamics defined this way splits in a natural way into a superposition of the "kinetic" evolution and the "potential" evolution. When the time-spacing τ tends to zero, the Trotter formula gives us the (spatially discretized) Hamiltonian, which is a sum of the kinetic and the potential energy of the field.

Mathematically, our construction is based on a lattice approximation for gauge fields (see [8,10]) where, like in the continuum theory (and unlike in Wilson's approach, cf. [18]), the gauge potential A is described by Lie algebra (not Lie group) elements assigned to the one-dimensional elements ("links") of the lattice. As a consequence, also the electric induction field D_k becomes a self-adjoint operator with continuous spectrum – like in the continuous version of the theory. However, the "longitudinal part" of the electric induction field, corresponding to electric charges, should have a discrete ("quantized") spectrum. In our construction this phenomenon is due to the gauge invariance of the physical wave functions. As will be seen in Section 4, the gauge invariance implies the compactification of some degrees of freedom of the field. They are no longer described by an $\mathbb{R}^1$-variable but by

a U(1)-variable. This non-trivial topology of the reduced configuration space is responsible for the quantization of the corresponding conjugate momenta, which turn out to be exactly electric charges, carried by the matter field. Reducing consequently both the Hilbert space of quantum states and the Feynman kernel with respect to the gauge group, we end up with the gauge-invariant version of the theory, described essentially in terms of the quantities arising in its continuous version (cf. results obtained in [14,15] for the continuous, spinorial electrodynamics).

The paper is organized as follows. In Section 1 we define the model. In Section 2 we present our lattice approximation and reduce it with respect to the gauge group. In Section 3 we show how the classical dynamics of the model may be completely described in terms of a complete system of gauge invariant observables. Also, we prove the causality of the dynamics. In Section 4 we construct the Hilbert space of (gauge-invariant) quantum states of the field, i.e. we find the representation of the reduced canonical commutation relations. In Section 5 we derive the quantum dynamics of the model from the gauge-invariant version of the Feynman integral, with few more technical calculations postponed to Section 6. Finally, in Section 7 we give an equivalent formulation of the theory, in terms of the configuration variables only, without any use of the canonical momenta.

We consider the present result as an introductory step towards the construction of the exact spinor electrodynamics on a lattice, which we are going to present in another paper.

1. Continuous version of the theory

1.1. Definition of the model

We consider the theory of a complex-valued matter field ϕ interacting with the electromagnetic field represented by the potential A_μ. The theory is defined by the Lagrangian

$$\mathcal{L} = -V(|\phi|^2) - \tfrac{1}{2}\overline{D_\mu\phi}D^\mu\phi - \tfrac{1}{4}f_{\mu\nu}f^{\mu\nu}, \tag{1.1}$$

where

$$D_\mu\phi = \partial_\mu\phi + i\,\frac{e}{\hbar}\,A_\mu\phi, \qquad f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.2}$$

and V is a function of a real variable (e.g. the Higgs potential). To simplify the formulae we will denote by $g = e/\hbar$ the electromagnetic coupling constant. The metric tensor is equal to $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The gauge group U(1) acts on the space of field configurations as follows:

$$\tilde{\phi} = e^{-i\lambda}\phi, \qquad \tilde{A}_\mu = A_\mu + (1/g)\partial_\mu\lambda. \tag{1.3}$$

Variation of the Lagrangian (1.1) with respect to the matter field leads to the second-order, non-linear Klein–Gordon equation

$$D_\mu D^\mu\phi - 2\,V'\phi = 0 \tag{1.4}$$

for the matter field, whereas variation with respect to the gauge potential gives the Maxwell equation (we use Gauss system of units).

$$\partial_\nu f^{\mu\nu} = -g \, \text{Im} \left(\bar{\phi} \, D^\mu \phi \right), \tag{1.5}$$

with the right-hand side being the electric current j^μ carried by the matter field ϕ.

1.2. Gauge freedom reduction

Configurations which differ only by a gauge transformation are physically equivalent. To describe the "true degrees of freedom" of the field we have to pass to the quotient space with respect to this equivalence. This may be done, e.g. by a gauge-fixing. For our purposes we use, however, a different description, namely a description in terms of the so-called hydrodynamical invariants (see [3,11,8])

$$R = |\phi|, \qquad v_\mu = \frac{1}{ig} \frac{\bar{\phi}}{|\phi|} D_\mu \frac{\phi}{|\phi|} = A_\mu + \frac{1}{g} \partial_\mu (\arg \phi). \tag{1.6}$$

Inserting these invariants in Eqs. (1.4) and (1.5) leads to the following field equations:

$$\partial_\mu \partial^\mu R - 2V' R = g^2 v_\mu v^\mu R, \qquad \partial_\nu f^{\mu\nu} = -g^2 R^2 v^\mu. \tag{1.7}$$

The term "hydrodynamical invariants" is justified by the fact that (1.7) may be interpreted as the equation of motion for a charged fluid of density R^2 and velocity v^μ, having a non-standard constitutive equation which is implied uniquely by the function V (see [3]).

For field configurations with non-vanishing ϕ, the values of the hydrodynamical invariants enable us to reconstruct the field configuration (ϕ, A_μ) uniquely up to a gauge transformation and the field equations (1.7) are equivalent to the original equations. For a generic configuration, vanishing of ϕ along a two-dimensional world sheet implies that the field v_μ carries a *vortex*, i.e. the singularity of the curl of v (see [3,11]). The vortex has to fulfill the Dirac's quantization condition (see [6]). Under this condition, the reconstruction (up to a gauge transformation) of the field configuration from the invariants is, again, possible. There is a conservation law for the vortices. Hence, describing field dynamics, we have to take into account the "string-like" degrees of freedom carried by the vortices. This is relatively difficult in the continuous version of the theory, but may be done in a completely satisfactory way in its discretized version.

1.3. Phase space of initial data

Given a Cauchy hypersurface $\{x^0 = \text{const.}\}$, the initial data for the theory are described by the values of the fields ϕ, A_μ and their conjugate momenta

$$\pi^0 = 2 \frac{\partial \mathcal{L}}{\partial \{\partial_0 \bar{\phi}\}} = -D^0 \phi, \qquad p^{0\mu} = \frac{\partial \mathcal{L}}{\partial \{\partial_0 A_\mu\}} = -f^{0\mu} = f_{0\mu} \tag{1.8}$$

on the hypersurface. The factor 2 in the definition of π^0 appears, because

$$\pi^0 = \text{Re} \, \pi^0 + i \, \text{Im} \, \pi^0,$$

where

$$\operatorname{Re} \pi^0 = \frac{\partial \mathcal{L}}{\partial (\partial_0 \operatorname{Re} \phi)}$$

is the momentum conjugate to $\operatorname{Re} \phi$ and analogously for the imaginary part.

Vanishing of the momentum p^{00} (Dirac's *primary constraint*) implies the reduction of the phase space with respect to A_0. This way we obtain the reduced phase space described by the quantities $(\phi, A_k, \pi^0, p^{0k})$, $k = 1, 2, 3$. There remain, however, secondary constraints, implied by the zero component of the field equation (1.5):

$$(1/g)\partial_k p^{0k} - \operatorname{Im} \left(\bar{\pi}^0 \phi \right) = 0 . \tag{1.9}$$

It is easy to see that the above constraint is equivalent to the statement that initial data, which differ only by a gauge transformation on the Cauchy hyperplane are equivalent. The naive proof of this statement consists in reparameterizing the initial configurations (ϕ, A_k) by the invariants (R, v_k) and the phase of the matter field $\alpha := \arg \phi$. This is a point transformation in the (infinite dimensional) symplectic space of Cauchy data. The corresponding transformation for the momenta can be obtained by rewriting the pre-symplectic form in terms of new variables (cf. [5,9]). Using (1.6) and integrating by parts we obtain this way

$$\begin{aligned}
&(\operatorname{Re} \pi^0)\delta(\operatorname{Re} \phi) + \operatorname{Im}(\pi^0)\delta(\operatorname{Im} \phi) + p^{0k}\delta A_k \\
&= \tfrac{1}{2}(\bar{\pi}^0\delta\phi + \pi^0\delta\bar{\phi}) + p^{0k}\delta A_k \\
&= \left(\frac{1}{g}\partial_k p^{0k} - \operatorname{Im}(\bar{\pi}^0\phi) \right) \delta\alpha + \operatorname{Re}(\bar{\pi}^0\phi)\frac{\delta R}{R} + p^{0k}\delta v_k - \frac{1}{g}\partial_k (p^{0k}\delta\alpha).
\end{aligned} \tag{1.10}$$

The last (boundary) term vanishes when integrated over the entire Cauchy hyperplane. Hence, the gauge-invariants $(p^{0k}, \operatorname{Re}(\bar{\pi}^0\phi))$ are the momenta canonically conjugate to the invariants $(v_k, \log R)$, whereas the vanishing left-hand side of (1.9) becomes the momentum canonically conjugate to the (completely arbitrary) phase of the matter field ϕ. The secondary reduction yields therefore the phase space described by the gauge-invariant, unconstrained observables $(p^{0k}, \operatorname{Re}(\bar{\pi}^0\phi), v_k, \log R)$. Knowing them, it is again possible to reconstruct uniquely (up to a gauge transformation) the entire non-reduced initial data $(p^{0k}, \pi^0, A_k, \phi)$.

The above, naive picture may become a rigorous result if we also describe the topological degrees of freedom carried by vortices corresponding to zeros of the matter field ϕ (see [11] for the case of electrodynamics and [12,13] for the case of a non-abelian gauge group). In the present paper we construct the lattice version of this rigorous result.

2. Lattice approximation

2.1. Structure of the lattice

We construct the approximation of this theory on a four-dimensional lattice Λ in Minkowski space. We recall that the metric tensor is $\operatorname{diag}(-1, 1, 1, 1)$. As Λ, we take the

Cartesian product of the three-dimensional, cubic lattice Σ and the discrete time axis T. Thus, the lattice sites are the points $x = (\tau n^0, an^1, an^2, an^3) \in \mathbb{R}^4$, where τ denotes the lattice spacing in the time direction, a denotes the lattice spacing in the space directions and $n = (n^0, n^1, n^2, n^3) \in \mathbb{Z}^4$ is a point with integer coordinates. Sites of Λ will be denoted by $x, \ldots$, links $(x, x + \hat{\mu}), \ldots$, plaquettes $(x; \hat{\mu}, \hat{\nu}), \ldots$, where $\hat{\mu}$ is a vector of length $a(\mu)$ (i.e. τ for $\mu = 0$ and a for the space dimensions) and direction of the oriented μth axis. We adopt the following conventions for summation ranges:

$$\mu, \nu = 0, 1, 2, 3, \qquad k, l = 1, 2, 3,$$
$$\hat{\mu}, \hat{\nu} = \hat{0}, -\hat{0}, \hat{1}, -\hat{1}, \hat{2}, -\hat{2}, \hat{3}, -\hat{3}, \qquad \hat{k}, \hat{l} = \hat{1}, -\hat{1}, \hat{2}, -\hat{2}, \hat{3}, -\hat{3}.$$

We will begin with a spatially bounded lattice Σ. The evolution of the field over such a lattice is not uniquely determined by the evolution equations unless we fix boundary conditions for the field in an appropriate way. Here, we do not impose any non-physical (e.g. cyclic) boundary conditions but we only fix the value of the field on the external sites and links of Σ.

There is a natural, inductive relation between the theories obtained for Σ_1 and Σ_2, when $\Sigma_1 \subset \Sigma_2$. Using this relation, as a last step of the construction of the field theory, we may pass to the inductive (thermodynamic) limit and construct this way the corresponding theory for the complete (unbounded) lattice $\mathbb{Z}^4$.

2.2. Field configurations on the lattice

We will represent the fields (ϕ, A_μ) on the lattice in the following way. The matter field will be represented by its complex value ϕ_x at every site x. The gauge field will be represented by the mean value $A_{x,x+\hat{\mu}} \in u(1) = \mathbb{R}^1$ of the component A_μ of the electromagnetic potential, on the link $(x, x + \hat{\mu})$.

The gauge group of the theory is described by real-valued functions of lattice sites: $\lambda_x \in u(1) = \mathbb{R}^1$. A gauge transformation is defined as a discrete version of (1.3):

$$\tilde{\phi}_x = e^{-i\lambda_x} \phi_x, \qquad \tilde{A}_{x,x+\hat{\mu}} = A_{x,x+\hat{\mu}} + \frac{\lambda_{x+\hat{\mu}} - \lambda_x}{ga(\mu)}. \tag{2.1}$$

The description of the gauge potentials and gauge transformations in terms of elements of the Lie algebra $u(1) = \mathbb{R}^1$ and not the Lie group $U(1)$, has been proposed in [10]. Heuristically, such a description was motivated by the geometric structure of the theory: for each lattice link $(x, x + \hat{\mu})$ it is possible to choose an internal gauge transformation (i.e. a gauge which is trivial on the link's ends) which makes the corresponding A_μ constant on the link. In the lattice approximation of the theory we want to keep this property. Hence, we describe the gauge field by this constant value A_μ, instead of replacing it by its truncated value $e^{iga(\mu)A_\mu}$.

To remain consistent with the above picture, gauge transformations on the lattice have to be considered as restrictions of continuous gauge transformations to the lattice sites. Therefore, it *does* make a difference whether we gauge the field at the point x by 0 or

by 2π, because, after interpolation of the transformation to the entire link $(x, x + \hat{\mu})$, the resulting value of $A_{x,x+\hat{\mu}}$ will be transformed in a different way.

We define the following lattice expressions for the covariant derivative $D_\mu\phi$ of the matter field and for the curl $f_{\mu\nu}$ of the gauge field:

$$(D\phi)_{x,x+\hat{\mu}} = \frac{1}{a(\mu)}\left(\exp\left(iga(\mu)A_{x,x+\hat{\mu}}\right)\phi_{x+\hat{\mu}} - \phi_x\right),$$

$$f_{x;\hat{\mu},\hat{\nu}} = \frac{1}{a(\mu)a(\nu)}\left(a(\mu)A_{x,x+\hat{\mu}} + a(\nu)A_{x+\hat{\mu},x+\hat{\mu}+\hat{\nu}}\right. \tag{2.2}$$

$$\left. + a(\mu)A_{x+\hat{\mu}+\hat{\nu},x+\hat{\nu}} + a(\nu)A_{x+\hat{\nu},x}\right).$$

It follows immediately that f, being the lattice curl of A, is invariant with respect to gauge transformations (2.1) and that $(D\phi)_{x,x+\hat{\mu}}$ behaves like ϕ_x under these transformations.

2.3. Reduction of gauge freedom

Physical field configurations are described by the space of gauge orbits, i.e. the quotient space of the above field configurations modulo gauge transformations. We will use different, equivalent parameterizations of this quotient space. Below we give a list of four such parameterizations. In fact, we need only the first and the last of them. The first parameterization is closely related to the gauge-dependent parameters (ϕ, A). The last one gives the final set of parameters, which we use to describe the theory. The easiest way to pass from one to the other (e.g. in the functional integral) is to use the remaining two parameterizations as intermediate steps.

Parameterization 1 ("tree-gauge"). If there is a gauge-condition such that each class of gauge-equivalent configurations contains one and only one representative fulfilling this condition, the entire class can be parameterized by this particular representative. A convenient way to fix a gauge consists in fixing the values $A_{x,x+\hat{\mu}}$ of the gauge field for all the links $(x, x + \hat{\mu})$ belonging to a set $\mathcal{T}$ of links which has properties of a *tree* (see [12]). By a *tree* we mean a pair $(x_0, \mathcal{T})$, where $x_0 \in \Lambda$ is a lattice site which we call a *root* of the tree and $\mathcal{T}$ is a subset of links of the lattice, having the following property: for any site $x \in \Lambda$ there is one and only one path connecting x with the root x_0 and composed of links belonging to the tree. We denote this path by $(x_0, x)_\mathcal{T}$.

By a tree-gauge we mean fixing the value of all the gauge potentials *along the tree* (e.g. putting them equal to zero). Now we may use all the remaining parameters (i.e. the values of the matter field ϕ_x and the values of the gauge potential $A_{x,x+\hat{\mu}}$ on all the links which *do not belong* to the tree) as parameters in the space of gauge orbits. There is, however, a residual gauge which still remains: the global gauge transformation which rotates all the values ϕ_x in the same way and does not change the values of the gauge potential. To remove this freedom we may fix the phase of the matter field at the root (assuming e.g. that ϕ_{x_0} is real).

A most simple example of a tree can be obtained as follows. Choose any root and take as $\mathcal{T}$ the following collection of links:

(1) all the links $(x, x + \hat{3})$ belonging to the x^3-axis passing through the root,
(2) all the links $(x, x + \hat{2})$, belonging to the two-dimensional plane (x^2, x^3) passing through the root,
(3) all the links $(x, x + \hat{1})$ belonging to the three-dimensional plane (x^1, x^2, x^3) passing through the root, and finally,
(4) all the time-like links $(x, x + \hat{0})$.

Fixing A_0 is usually called a *temporal gauge*. This is however not a complete gauge condition, since any gauge transformation which is time-independent leaves the condition unchanged. Fixing also gauge fields on the remaining space-like links of the above tree and fixing the phase of the matter field at the root, removes this residual gauge-freedom.

A three-dimensional tree defined by (1)–(3) can also be used to fix completely a gauge in the space of Cauchy data on a fixed three-dimensional lattice-layer $\{x^0 = \text{const.}\}$. This is a generalization of an axial gauge-condition.

Parameterization 2 ("improved tree-gauge"). Keeping a tree-gauge, we may parameterize the electromagnetic degrees of freedom by the values of the field strength f. However, not all the values of $f_{x;\hat{\mu},\hat{\nu}}$ are independent, because f is a closed form. This means that the sum of f over all the plaquettes belonging to the boundary of any three-dimensional cube vanishes. To find independent parameters we take for every *off-tree* link $(x, x + \hat{\mu})$ the closed path composed of (1) the tree-path $(x_0, x)_\mathcal{T}$, (2) the link $(x, x + \hat{\mu})$ itself and (3) the inverse tree-path $(x + \hat{\mu}, x_0)_\mathcal{T}$. The above closed path is the boundary of a surface composed of a finite number of "along-tree-plaquettes". Take the value $\mathcal{F}_{x,x+\hat{\mu}}(\mathcal{T})$ of the electromagnetic flux through it, i.e. the sum of all the values $f_{x;\hat{\mu},\hat{\nu}}$ corresponding to this surface. The collection of fluxes $\mathcal{F}_{x,x+\hat{\mu}}(\mathcal{T})$ corresponding to all the *off-tree* links contains the complete, independent information about the gauge field. To prove this statement we observe that $\mathcal{F}_{x,x+\hat{\mu}}(\mathcal{T})$ is equal to the value of $a(\mu)A_{x,x+\hat{\mu}}$ plus the sum of all the *along-tree* values of the gauge potential A (multiplied by the length of the corresponding link), over the entire path $(x_0, x)_\mathcal{T}$ and $(x + \hat{\mu}, x_0)_\mathcal{T}$. The latter being fixed by the tree gauge, we see that the information contained in $\mathcal{F}_{x,x+\hat{\mu}}(\mathcal{T})$ is equivalent to the information about $A_{x,x+\hat{\mu}}$. We conclude that the values of f on the plaquettes containing tree-links ("along-tree-plaquettes") can be chosen as independent variables.

Parameterization 3 ("long hydrodynamical invariants"). For a given field configuration $(\phi_x, A_{x,x+\hat{\mu}})$ on the lattice, we define the lattice analog of hydrodynamical invariants (1.6):

$$R_x = |\phi_x|, \qquad v_{x,x+\hat{\mu}} = A_{x,x+\hat{\mu}} + \frac{\alpha_{x+\hat{\mu}} - \alpha_x}{ga(\mu)}, \tag{2.3}$$

where α_x is *a phase* of ϕ_x, i.e. any real number satisfying the equation

$$R_x \, e^{i\alpha_x} = \phi_x. \tag{2.4}$$

Since the phase is defined up to $2n\pi$, the above quantities are defined only up to a residual gauge transformation

$$\bar{v}_{x,x+\hat{\mu}} = v_{x,x+\hat{\mu}} + \frac{\lambda_{x+\hat{\mu}} - \lambda_x}{ga(\mu)}, \tag{2.5}$$

with the gauge function assuming discrete values only: $\lambda_x \in 2\pi\mathbb{Z}$. The value of the electromagnetic field is equal to the curl of v:

$$f_{x;\hat{\mu},\hat{\nu}} = \frac{1}{a(\mu)a(\nu)}\left(a(\mu)v_{x,x+\hat{\mu}} + a(\nu)v_{x+\hat{\mu},x+\hat{\mu}+\hat{\nu}}\right.$$
$$\left. + a(\mu)v_{x+\hat{\mu}+\hat{\nu},x+\hat{\nu}} + a(\nu)v_{x+\hat{\nu},x}\right). \tag{2.6}$$

Sometimes it is useful to exclude from the configuration space the zero-measure set of configurations with vanishing values of R. To parameterize the remaining configurations we may use $\rho_x = \log R_x$. The space of values of all variables ρ and v describing the initial data is topologically equivalent to $\mathbb{R}^{N+L}$, where N is the number of sites and L is the number of links contained in Σ. We stress however that there is still the residual (discrete) gauge transformation (2.5) acting in this space. The corresponding group is topologically equivalent to $\mathbb{Z}^N$. The space of initial configurations can be obtained as the quotient with respect to this group action, i.e. it is equivalent to $\mathbb{R}^L \times T^N$ where by T^N we denote the N-dimensional torus. On the classical level this fact has no specific consequences. We will however see that in the quantum version of the theory this non-trivial topology will be responsible for the electric charge quantization.

Parameterization 4 ("short hydrodynamical invariants"). Due to the above transformation properties of $v_{x,x+\hat{\mu}}$, the following U(1)-valued object is a genuine gauge invariant:

$$W_{x,x+\hat{\mu}} = \exp(iga(\mu)v_{x,x+\hat{\mu}})$$
$$= \left(\phi_x R_x^{-1}\right)^{-1} \exp(iga(\mu)A_{x,x+\hat{\mu}}) \left(\phi_{x+\hat{\mu}} R_{x+\hat{\mu}}^{-1}\right). \tag{2.7}$$

Obviously, we have

$$W_{x+\hat{\mu},x} = W_{x,x+\hat{\mu}}^{-1} = W_{x,x+\hat{\mu}}^*.$$

Passing from v's to W's we partially have lost information about the electromagnetic field. Indeed, due to equation

$$W_{x,x+\hat{\mu}} \cdot W_{x+\hat{\mu},x+\hat{\mu}+\hat{\nu}} \cdot W_{x+\hat{\mu}+\hat{\nu},x+\hat{\nu}} \cdot W_{x+\hat{\nu},x} = \exp\left(iga(\mu)a(\nu)f_{x;\hat{\mu},\hat{\nu}}\right) \tag{2.8}$$

we are able to reconstruct $f_{x;\hat{\mu},\hat{\nu}}$ only up to $2n\pi/ga(\mu)a(\nu)$. The complete gauge-invariant information about the field configuration is therefore given by the following set of invariants: $\left(R_x, W_{x,x+\hat{\mu}}, f_{x;\hat{\mu},\hat{\nu}}\right)$, where $R_x > 0$, $|W_{x,x+\hat{\mu}}| = 1$ and f is a closed two-form, fulfilling (2.8).

To choose a maximal set of independent parameters among all the above ones, we may again use any tree $\mathcal{T}$ and to select the set composed of the following invariants:
(1) all the values R_x,
(2) all the "along-tree" values $W_{x,x+\hat{\mu}}$,
(3) all the fluxes $\mathcal{F}_{x,x+\hat{\mu}}(\mathcal{T})$ corresponding to all the "off-tree" links.

We see that the compactified (U(1)-valued) degrees of freedom are now described by the quantities $W_{x,x+\hat{\mu}}$, whereas the remaining degrees of freedom have trivial topology.

A similar parameterization may be used to describe in a gauge-independent way the space of Cauchy data on a three-dimensional layer of the lattice given by $\{x^0 = \text{const.}\}$, but here only space-like μ's are necessary. The electromagnetic field on space-like plaquettes reduces to the magnetic field, $f_{x;\hat{k},\hat{l}} = B_{x;\hat{k},\hat{l}}$ and we have

$$W_{x,x+\hat{k}} \cdot W_{x+\hat{k},x+\hat{k}+\hat{l}} \cdot W_{x+\hat{k}+\hat{l},x+\hat{l}} \cdot W_{x+\hat{l},x} = \exp\left(\mathrm{i}ga^2 B_{x;\hat{k},\hat{l}}\right). \tag{2.9}$$

Since B is a closed form, for any three-dimensional cube the following combination vanishes identically:

$$B_{x;\hat{k},\hat{l}} + B_{x;\hat{n},\hat{k}} + B_{x;\hat{l},\hat{n}} + B_{x+\hat{n};\hat{l},\hat{k}} + B_{x+\hat{l};\hat{k},\hat{n}} + B_{x+\hat{k};\hat{n},\hat{l}} \equiv 0 . \tag{2.10}$$

Again, as independent variables we may take only "along-tree" values of $W_{x,x+\hat{k}}$ and only "off-tree" values of the magnetic flux $\mathcal{F}_{x,x+\hat{k}}(T)$, for any fixed three-dimensionaltree T.

2.4. Reconstruction of the field configuration from invariants

To reconstruct the field configuration $(\phi_x, A_{x,x+\hat{\mu}})$ from the "long hydrodynamical invariants" $(R_x, v_{x,x+\hat{\mu}})$ we choose any tree (x_0, T) and put an arbitrary value of $A_{x,x+\hat{\mu}}$ along the tree. We may also choose the phase α_{x_0} at the tree-root in an arbitrary way. Moving along the tree and using the definition (2.3) of v we may now reconstruct all the phases α_x at all the sites. Then, we reconstruct uniquely the matter field ϕ_x from (2.4) and the remaining gauge potentials $A_{x,x+\hat{\mu}}$ from (2.3).

To reconstruct the field configuration from the "short hydrodynamical invariants" it is, therefore, sufficient to reconstruct the long invariants v. Choose first any values $\tilde{v}$ satisfying (2.7) and define $\tilde{f}$ as the lattice curl of $\tilde{v}$. The lattice two-form $(f - \tilde{f})$ is closed and integer-valued (more precisely, Eq. (2.8) implies that it takes values in the set $(2\pi/ga(\mu)a(\nu))\mathbb{Z}$). Using the lattice version of the Poincaré lemma we see that there exists an integer-valued one-form β on the lattice such that $\text{curl}\,\beta = f - \tilde{f}$. Hence, the one-form $v := \tilde{v} + \beta$ fulfills (2.6) and may be taken as a possible representative of long hydrodynamical parameters.

3. Classical field dynamics

3.1. Lagrangian formulation of dynamics

To obtain the difference equations describing the dynamics of the field on the lattice we use the following approximation of the Lagrangian (1.1):

$$\mathcal{L}_x = -V(|\phi_x|^2) - \frac{1}{2}\sum_\mu g^{\mu\mu}\left|(D\phi)_{x,x+\hat{\mu}}\right|^2 - \frac{1}{4}\sum_{\mu,\nu} g^{\mu\mu}g^{\nu\nu}\left(f_{x;\hat{\mu},\hat{\nu}}\right)^2 , \tag{3.1}$$

where we sum over positive axes directions only. The action for a given region $\mathcal{V}$ of space–

time is given by

$$S_V = \sum_{x \in V} \tau a^3 \mathcal{L}_x \, , \tag{3.2}$$

where we sum only over the internal sites. As we already mentioned, the fields on the boundary sites and links have to be fixed. They are used as the boundary conditions for the dynamics. The dynamics is deduced from the above action and is given by the following system of second-order difference equations:

$$
\begin{aligned}
0 &= \frac{\partial S}{\partial \bar{\phi}_x} = -\frac{1}{2} \tau a^3 \left(2V' \cdot \phi_x - \sum_{\hat{\mu}} \frac{g^{\mu\mu}}{a(\mu)} (D\phi)_{x,x+\hat{\mu}} \right) , \\
0 &= \frac{\partial S}{\partial A_{x,x+\hat{\mu}}} = -g^{\mu\mu} \tau a^3 \left(g \, \mathrm{Im}\big(\bar{\phi}_x (D\phi)_{x,x+\hat{\mu}}\big) + \sum_{\hat{v}} \frac{g^{vv}}{a(v)} f_{x;\hat{\mu},\hat{v}} \right) .
\end{aligned}
\tag{3.3}
$$

Introducing the canonical momenta, analogous to (1.8),

$$\pi_x^{\hat{\mu}} = -g^{\mu\mu}(D\phi)_{x,x+\hat{\mu}}, \qquad p_x^{\hat{\mu}\hat{v}} = -g^{\mu\mu} g^{vv} f_{x;\hat{\mu},\hat{v}}, \tag{3.4}$$

the same equations may be written as a system of first-order difference equations

$$-2V' \cdot \phi_x = \sum_{\hat{\mu}} \frac{1}{a(\mu)} \pi_x^{\hat{\mu}}, \qquad -g \, \mathrm{Im}\big(\bar{\phi}_x \, \pi_x^{\hat{\mu}}\big) = \sum_{\hat{v}} \frac{1}{a(v)} p_x^{\hat{\mu}\hat{v}}. \tag{3.5}$$

3.2. Canonical momenta

Initial data for the theory consist of field configurations and the corresponding momenta in a single three-dimensional lattice layer Σ_t, given by $\{x^0 = t\}$. Because momenta (3.4) are always attached to the time-like lattice links and plaquettes, we may choose between the future-directed and the past-directed momenta. In the first case the complete Cauchy data are given by the collection $(\phi_x, A_{x,x+\hat{k}}, \pi_x^{\hat{0}}, p_x^{\hat{0}\hat{k}})$. To keep the future orientation of the momenta, we will describe the Cauchy data in the second case by $(\phi_x, A_{x,x+\hat{k}}, -\pi_x^{-\hat{0}}, -p_x^{-\hat{0}\hat{k}})$. We will call the first collection "the Cauchy data on Σ_t^+" and the second one "on Σ_t^-". We may imagine Σ_t^+ (resp. Σ_t^-) as the hypersurface obtained from Σ_t by a tiny translation in the positive (resp. negative) direction of the time axis.

To simplify the notation we will denote

$$
\begin{aligned}
\pi_x^+ &:= \pi_x^{\hat{0}}, & \pi_x^- &:= -\pi_x^{-\hat{0}}, \\
D_{x,x+\hat{k}}^+ &:= -p_x^{\hat{0}\hat{k}} = -f_{x;\hat{0},\hat{k}}, & D_{x,x+\hat{k}}^- &:= p_x^{-\hat{0}\hat{k}} = f_{x;-\hat{0},\hat{k}},
\end{aligned}
$$

where by D we denote the electric components of the electromagnetic field (the magnetic components are already denoted by B). Notice that, similarly as in the continuous version of the theory, the momentum canonically conjugate to the gauge potential A_k is given by *minus* the electric induction field. This means that the momentum conjugate to $A_{x,x+\hat{k}}$ on Σ^+ (resp. Σ^-) is given by $-D_{x,x+\hat{k}}^+$ (resp. $-D_{x,x+\hat{k}}^-$).

There is, however, an additional factor a^3, which appears in the canonical structure of the Cauchy data. Indeed, in the continuous version of the theory the canonical structure is given by the following Poisson bracket:

$$\left\{ \operatorname{Re} \pi(x), \operatorname{Re} \phi(y) \right\} = \delta^{(3)}(x - y)$$

and similar formulae for the remaining degrees of freedom. Integrating the above formula over a domain $\mathcal{V} \subset \Sigma$ with respect to the variable x we obtain

$$\left\{ \int_{\mathcal{V}} \operatorname{Re} \pi(x) \, \mathrm{d}^3 x, \operatorname{Re} \phi(y) \right\} = \delta_{\mathcal{V}, y} := \begin{cases} 1 & \text{when } y \in \mathcal{V}, \\ 0 & \text{otherwise.} \end{cases}$$

In the lattice version of the theory the discretized observables represent their mean values over elementary cubes. Hence, we should choose as $\mathcal{V}$ such an elementary cube and replace the integral by $a^3 \operatorname{Re} \pi$. This way we obtain the canonical Poisson bracket:

$$\left\{ a^3 \operatorname{Re} \pi_x, \operatorname{Re} \phi_y \right\} = \delta_{x, y} .$$

We conclude that the momentum canonically conjugate to ϕ_x is equal to $a^3 \bar{\pi}_x$, and the momentum canonically conjugate to $A_{x, x+\hat{k}}$ is equal to $-a^3 D_{x, x+\hat{k}}$.

3.3. Initial value problem

The transition from Σ_t^- to Σ_t^+ will be called *potential evolution*. The transition from Σ_t^+ to $\Sigma_{t+\tau}^-$ will be called *kinetic evolution*. The global evolution of the field is given by the successive composition of the potential and kinetic evolution operators.

As a result of the potential evolution, the field configurations $(\phi_x, A_{x, x+\hat{k}})$ remain unchanged and the momenta change their value from $(\pi_x^-, D_{x, x+\hat{k}}^-)$ to $(\pi_x^+, D_{x, x+\hat{k}}^+)$ according to formulae (3.5). We have therefore

$$\pi_x^+ = \pi_x^- - 2\tau \mathrm{V}' \cdot \phi_x + \frac{\tau}{a} \sum_{\hat{k}} (D\phi)_{x, x+\hat{k}} \tag{3.6}$$

and

$$D_{x, x+\hat{k}}^+ = D_{x, x+\hat{k}}^- + g\tau \, \operatorname{Im} \left(\bar{\phi}_x (D\phi)_{x, x+\hat{k}} \right) + \frac{\tau}{a} \sum_{\hat{l}} B_{x; \hat{k}, \hat{l}} . \tag{3.7}$$

During the kinetic evolution the momenta are transported parallelly:

$$\pi_{x+\hat{0}}^- = \exp \left(-\mathrm{i} g\tau A_{x, x+\hat{0}} \right) \pi_x^+ , \tag{3.8}$$

and

$$D_{x+\hat{0}, x+\hat{0}+\hat{k}}^- = D_{x, x+\hat{k}}^+ , \tag{3.9}$$

whereas the configurations undergo the linear evolution according to formulae (3.4). This implies for the matter field the following evolution:

$$\exp \left(\mathrm{i} g\tau A_{x, x+\hat{0}} \right) \phi_{x+\hat{0}} = \phi_x + \tau \pi_x^+ .$$

Hence, the kinetic evolution is given uniquely *up to a gauge transformation*. Indeed, fixing in an arbitrary way the value of the gauge field on time-like links we may derive uniquely the final data from the initial data. Putting e.g. $A_{x,x+\hat{0}} = 0$ we obtain from (3.4):

$$\phi_{x+\hat{0}} = \phi_x + \tau \pi_x^+ , \tag{3.10}$$

$$A_{x+\hat{0},x+\hat{0}+\hat{k}} = A_{x,x+\hat{k}} - \tau D^+_{x,x+\hat{k}} , \tag{3.11}$$

whereas the momenta remain constant due to (3.8) and (3.9).

We stress that both the potential and the kinetic evolution are local: to find the final data in a bounded portion $\mathcal{V}$ of the lattice we do not need to know the entire initial data on Σ, but only a portion of it, contained in the discrete causal shadow of $\mathcal{V}$.

On quantum level the potential evolution will be generated by the potential part of the action (i.e. the sum of all the terms assigned to the lattice sites and to the purely space-like lattice links and plaquettes), whereas the kinetic evolution will be generated by the kinetic part of the action (i.e. the sum of the terms assigned to the time-like links and plaquettes). In the $\tau \to 0$ limit we obtain, due to the Trotter formula, the continuous evolution generated by the sum of the potential and the kinetic energy.

3.4. Gauge-invariant description of initial data

Due to the gauge properties of ϕ and π, their combination $\pi\bar{\phi}$ is gauge-invariant. Dividing it into the real part K and the imaginary part M we observe that the latter has to fulfill a constraint analogous to (1.9), implied by the zero component of (3.5):

$$\frac{1}{g}\left(\frac{1}{a}\sum_{\hat{k}} D^+_{x,x+\hat{k}}\right) - M_x^+ = 0 = \frac{1}{g}\left(\frac{1}{a}\sum_{\hat{k}} D^-_{x,x+\hat{k}}\right) - M_x^- , \tag{3.12}$$

where we denote

$$\pi_x^+ \bar{\phi}_x =: K_x^+ + iM_x^+ , \qquad \pi_x^- \bar{\phi}_x =: K_x^- + iM_x^- . \tag{3.13}$$

Similarly as in the continuous theory, we may reduce the phase space of Cauchy data on every Σ_t with respect to these constraints. As a result we obtain a formula analogous to (1.10) with the continuous divergence operator replaced by its lattice analog. We conclude that K (with the index "+" if we are on Σ^+ and "−" if we are on Σ^-) is the momentum canonically conjugate to $\rho = \log R$, whereas the electric induction field D is the momentum canonically conjugate to the long hydrodynamical variable v. The vanishing left- (resp. right-)hand side of (3.12) has to be considered as the momenta canonically conjugate to the (completely arbitrary) phase of the matter field. Knowing (R, v, K, D) on Σ we may first reconstruct the configuration (ϕ, A) (uniquely up to a gauge transformation). Then, we obtain the missing information M about the momentum π from the lattice divergence of the electric induction field D, according to (3.12). This way we reconstruct (uniquely up to a gauge transformation) the complete initial data (ϕ, A, π, D).

Splitting the matter field into two degrees of freedom corresponding to its real and imaginary part: $\phi = \phi_1 + i\phi_2$, the corresponding momentum decomposes accordingly into $\pi = \pi_1 + i\pi_2$. Thus

$$K = \phi_1\pi_1 + \phi_2\pi_2, \qquad M = \phi_1\pi_2 - \phi_2\pi_1 . \tag{3.14}$$

are the generators of the two-dimensional dilatations and rotations, respectively.

3.5. Time evolution in terms of gauge-invariants

Both the potential and the kinetic evolution may be rewritten in terms of the invariants. During the potential evolution the configurations (R, W, B) remain unchanged whereas the following equations for the momenta may be immediately deduced from (3.6), (3.7) and the definition (3.13) of K:

$$
\begin{aligned}
K^+_x &= K^-_x - 2\tau V'\left(R^2_x\right) \cdot R^2_x - \frac{\tau}{a^2}\sum_{\hat{k}}\left(R^2_x - R_x R_{x+\hat{k}} \ \mathrm{Re}\ W_{x,x+\hat{k}}\right), \\
D^+_{x,x+\hat{k}} &= D^-_{x,x+\hat{k}} + g\frac{\tau}{a}R_x R_{x+\hat{k}}\ \mathrm{Im}\ W_{x,x+\hat{k}} + \frac{\tau}{a}\sum_{\hat{l}}B_{x;\hat{k},\hat{l}} .
\end{aligned}
\tag{3.15}
$$

The last equation is the lattice version of the Maxwell equation $\dot{D} = -j + \mathrm{curl}\ B$.

The simplest way to rewrite the kinetic evolution in terms of invariants is to fix the temporal gauge $A_{x,x+\hat{0}} = 0$. With such a choice the kinetic evolution of the matter field over each time-like link becomes a free evolution of the two degrees of freedom described by ϕ. The evolution may be thought of as generated by the free Hamiltonian $a^3 H$, where

$$H = \tfrac{1}{2}\bar{\pi}\pi = \tfrac{1}{2}R^{-2}(K^2 + M^2)$$

is the Hamiltonian density and the factor a^3 comes from the integration over an elementary cube. It is obvious that H remains constant during the kinetic evolution. This implies that its value on both ends of the link is the same. Also the "angular momentum" M remains constant in agreement with formula (3.12) (remember that M is not an independent variable but has to be treated as the divergence of the electric induction field).

Calculating the Poisson bracket of various quantities with the above Hamiltonian we obtain their evolution along the link $(x, x + \hat{0})$. Hence,

$$\dot{K} = \{K, a^3 H\} = 2H \tag{3.16}$$

implies

$$K^-_{x+\hat{0}} = K^+_x + 2\tau H^+_x . \tag{3.17}$$

Moreover,

$$(R^2)^{\cdot} = \{R^2, a^3 H\} = 2(\phi_1\pi_1 + \phi_2\pi_2) = 2K \tag{3.18}$$

implies the evolution of R, compatible with (3.10):

$$R^2_{x+\hat{0}} = |\phi_{x+\hat{0}}|^2 = |\phi_x + \tau\pi^+_x|^2 = R^2_x + 2\tau K^+_x + 2\tau^2 H^+_x . \tag{3.19}$$

The lattice curl of Eq. (3.11) implies the evolution of the magnetic field, analogous to the Maxwell equation $\dot{B} = -\text{curl}\, D$:

$$B_{x+\hat{0};\hat{k}\hat{l}} = B_{x;\hat{k},\hat{l}} - \frac{\tau}{a}\left(D^+_{x,x+\hat{k}} + D^+_{x+\hat{k},x+\hat{k}+\hat{l}} + D^+_{x+\hat{k}+\hat{l},x+\hat{l}} + D^+_{x+\hat{l},x}\right). \tag{3.20}$$

To derive the kinetic evolution of W let us observe that the definition of D implies

$$W_{x+\hat{0},x+\hat{0}+\hat{k}} = W_{x+\hat{0},x}\exp\left(-\text{i}g\tau a D_{x,x+\hat{k}}\right)W_{x,x+\hat{k}}W_{x+\hat{k},x+\hat{0}+\hat{k}} . \tag{3.21}$$

Using the evolution of ϕ written in polar coordinates

$$\phi_{x+\hat{0}} = \phi_x + \tau\pi^+_x = \phi_x\left[1 + \tau R^{-2}_x\left(K^+_x + \text{i}M^+_x\right)\right],$$

we may express $W_{x,x+\hat{0}}$ in terms of Cauchy data on Σ^+ or on Σ^-:

$$\begin{aligned}
W_{x,x+\hat{0}} &= \left(\phi_x R^{-1}_x\right)^{-1}\phi_{x+\hat{0}}R^{-1}_{x+\hat{0}} \\
&= R_x\left[1 + \tau R^{-2}_x\left(K^+_x + \text{i}M^+_x\right)\right]\left(R^2_x + 2\tau K^+_x + 2\tau^2 H^+_x\right)^{-1/2} \\
&= \left(R^2_{x+\hat{0}} - 2\tau K^-_{x+\hat{0}} + 2\tau^2 H^-_{x+\hat{0}}\right)^{-1/2} \\
&\quad \times \left[1 - \tau R^{-2}_{x+\hat{0}}\left(K^-_{x+\hat{0}} - \text{i}M^-_{x+\hat{0}}\right)\right]R_{x+\hat{0}}.
\end{aligned} \tag{3.22}$$

Finally, substituting (3.22) into (3.21) we easily find the evolution of the "short" hydrodynamical variables,

$$\begin{aligned}
W_{x+\hat{0},x+\hat{0}+\hat{k}} &= \left(R^2_x + 2\tau K^+_x + 2\tau^2 H^+_x\right)^{-1/2}\left[1 + \tau R^{-2}_x\left(K^+_x - \text{i}M^+_x\right)\right] \\
&\quad \times R_x\exp(-\text{i}g\tau a D^+_{x,x+\hat{k}})W_{x,x+\hat{k}} \times R_{x+\hat{k}} \\
&\quad \times \left[1 + \tau R^{-2}_{x+\hat{k}}\left(K^+_{x+\hat{k}} + \text{i}M^+_{x+\hat{k}}\right)\right] \\
&\quad \times \left(R^2_{x+\hat{k}} + 2\tau K^+_{x+\hat{k}} + 2\tau^2 H^+_{x+\hat{k}}\right)^{-1/2} .
\end{aligned} \tag{3.23}$$

4. Quantum kinematics

4.1. Quantum states of the field

Let us begin with the naive Schrödinger representation of a quantum state Ψ as a wave function of field configurations:

$$\Psi = \Psi(\{\phi_x\}, \{A_{x,x+\hat{k}}\})$$

on our finite three-dimensional lattice Σ. The momenta canonically conjugate are represented

by derivatives with respect to configurations,

$$a^3(\text{Re }\pi_x) = \frac{\hbar}{i}\frac{\partial}{\partial(\text{Re }\phi_x)}\ , \qquad a^3(\text{Im }\pi_x) = \frac{\hbar}{i}\frac{\partial}{\partial(\text{Im }\phi_x)}$$

(similarly as in the classical case, the factor a^3 appears because of the integration of the densities of the above momenta over elementary cubes of the lattice). The above definitions may again be shortened as follows:

$$a^3\pi_x = 2\frac{\hbar}{i}\frac{\partial}{\partial\bar\phi_x}\ . \tag{4.1}$$

Similarly,

$$-a^3 D_{x,x+\hat{k}} = \frac{\hbar}{i}\frac{\partial}{\partial A_{x,x+\hat{k}}}\ . \tag{4.2}$$

This implies that the generator of two-dimensional rotations $M = \phi_1\pi_2 - \phi_2\pi_1$ (there is no ordering problem here!) is represented as the derivative with respect to the phase α_x of ϕ_x:

$$a^3 M_x = \frac{\hbar}{i}\frac{\partial}{\partial\alpha_x}\ . \tag{4.3}$$

To distinguish between the momenta on Σ^+ and Σ^- we will mark them, when necessary, with $^+$ or $^-$, respectively.

The physically meaningful quantum states have, however, to fulfill the Gauss constraint (3.12). Written in terms of the above momenta, the constraint implies the vanishing of the following operator:

$$G_x = -\frac{\partial}{\partial\alpha_x} - \frac{1}{ga}\sum_{\hat{k}}\frac{\partial}{\partial A_{x,x+\hat{k}}}\ . \tag{4.4}$$

But $\Sigma\lambda_x G_x$ is the generator of the gauge transformation (2.1). Wave functions which are annihilated by G_x are, therefore, those which are constant on gauge orbits. They cannot be square-integrable with respect to all the gauge parameters because gauge orbits are not compact. To define the quantized version of our theory we have to organize such functions in a Hilbert space. For this purpose we have to define a scalar product in the space of gauge-invariant wave functions. There have been many proposals how to define such a scalar product (cf. [1,2,16,17]). In this paper we shall use the product defined in [19,20] on geometrical grounds. However, we are not going to impose this definition a priori, but will derive it uniquely from the representation theorem for the algebra of observables.

4.2. Algebra of quantum observables

In the naive Schrödinger representation used above, quantum observables are those operators, which are gauge-invariant. The gauge-invariant functions of configurations belong to this class. They can be used as multiplication operators acting on gauge-invariant functions. This way we define quantum observables $(R_x, W_{x,x+\hat{k}}, B_{x;\hat{k},\hat{l}})$, where all R_x are self-adjoint

and positive, $W_{x,x+\hat{k}}$ are unitary and $B_{x;\hat{k},\hat{l}}$ are self-adjoint. All these operators commute with each other. They are not independent because they obviously fulfill the constraints (2.9) and (2.10).

Among the momenta, the electric induction field operator $D_{x,x+\hat{k}}$ given by (4.2) and the generator of the two-dimensional dilatation

$$K_x := \tfrac{1}{2}(\phi_1\pi_1 + \pi_1\phi_1 + \phi_2\pi_2 + \pi_2\phi_2) \tag{4.5}$$

(symmetrized version of the classical quantity (3.14)) produce a gauge-invariant function when applied to a gauge-invariant function. Hence, they are also observables. Denote by $\mathcal{O}_\Sigma$ the algebra generated by the above operators. The following commutation relations between the generators of $\mathcal{O}_\Sigma$ can be immediately checked:

$$\begin{aligned}
&\left[R_x, a^3 K_x\right] = i\hbar R_x , \\
&\left[W_{x,x+\hat{k}}, -a^3 D_{x,x+\hat{k}}\right] = -\hbar g a\, W_{x,x+\hat{k}} , \\
&\left[B_{x;\hat{k},\hat{l}}, -a^3 D_{x,x+\hat{k}}\right] = \frac{i\hbar}{a}\mathbf{1} ,
\end{aligned} \tag{4.6}$$

with all the remaining commutators vanishing.

Definition. The charge $Q_x \in \mathcal{O}_\Sigma$ at a lattice site x is defined as the lattice divergence of the electric induction field:

$$Q_x = a^3 \operatorname{div}_x D = a^2 \sum_{\hat{k}} D_{x,x+\hat{k}} = g a^3 M_x . \tag{4.7}$$

The commutation relation

$$[W_{x,x+\hat{k}}, (1/g)Q_x] = \hbar\, W_{x,x+\hat{k}} ,$$

follows immediately from the above definition of Q and from (4.6).

Lemma. *The operator $U_x := \mathrm{e}^{(2\pi i/e)Q_x}$ commutes with all the observables in $\mathcal{O}_\Sigma$.*

Proof. The operator Q_x is built of the operators D. Therefore, it commutes with all the observables R_y and K_y. Hence, we have to check the commutation of U with W and B. The commutator $[\operatorname{div}_x D, B_{x;\hat{k},\hat{l}}]$ vanishes, because it is a sum of the term $[D_{x,x+\hat{k}}, B_{x;\hat{k},\hat{l}}] = i\hbar/a^4$ and the term $[D_{x,x+\hat{l}}, B_{x;\hat{k},\hat{l}}] = -i\hbar/a^4$ (B is antisymmetric in $\hat{k}, \hat{l}$). To check the commutation with W consider a one-parameter family of operators

$$W^\lambda_{x,x+\hat{k}} = \mathrm{e}^{-i\lambda a^3 \operatorname{div}_x D} W_{x,x+\hat{k}} \mathrm{e}^{i\lambda a^3 \operatorname{div}_x D} .$$

Its derivative over λ equals

$$\partial W^\lambda_{x,x+\hat{k}}/\partial\lambda = \mathrm{e}^{-i\lambda a^3 \operatorname{div}_x D} \left[-i\, a^3 \operatorname{div}_x D\, W_{x,x+\hat{k}} + i\, W_{x,x+\hat{k}}\, a^3 \operatorname{div}_x D\right] \mathrm{e}^{i\lambda a^3 \operatorname{div}_x D}$$

and contains the commutator of $a^3 \operatorname{div} D$ and W equal to

$$a^2[D_{x,x+\hat{k}}, W_{x,x+\hat{k}}] = -g\hbar W_{x,x+\hat{k}} .$$

Hence,

$$\partial W^{\lambda}_{x,x+\hat{k}} / \partial\lambda = ig\hbar W^{\lambda}_{x,x+\hat{k}},$$

and consequently

$$W^{\lambda}_{x,x+\hat{k}} = e^{ig\hbar\lambda} W^{0}_{x,x+\hat{k}},$$

thus, for $\lambda_n = (2\pi/g\hbar)n$ with $n \in \mathbb{Z}$ we have $W^{\lambda_n} = W^0 = W$. This implies that the operator

$$e^{i\lambda_1 a^3 \operatorname{div}_x D} = e^{(2\pi i/g\hbar)Q_x}$$

commutes with all the observables, which ends the proof.   □

4.3. Existence and uniqueness of irreducible representation

Algebra $\mathcal{O}_{\Sigma}$ could be also defined without any use of the Schrödinger representation. For this purpose, classical gauge-invariant observables (R, W, B, D, K) can be taken, with their Poisson brackets replaced by commutators. Quantization of such a structure means a construction of its irreducible Hilbert space representation.

The Schrödinger representation in the space of all L^2-functions of field configurations ϕ_x and $A_{x,x+\hat{k}}$ is not irreducible, since there is a non-trivial, unitary operator U_x which commutes with all the observables. For irreducible representations, this commutativity would imply

$$U_x = e^{2\pi i\theta_x} \cdot 1.$$

The electric charge quantization would be an immediate consequence of the above condition. Indeed, the spectrum of Q_x covers the set $\{g\hbar \cdot n + \theta_x, n \in \mathbb{Z}\}$. Hence, $e := g\hbar$ plays the role of the elementary charge.

There are, however, non-equivalent irreducible representations corresponding to different values of the angle θ_x. To avoid non-symmetry of the charge spectrum we will always impose condition

$$\theta_x \equiv 0, \tag{4.8}$$

which implies already the uniqueness of the representation.

To prove this statement let us first construct a representation by introducing the following scalar product:

Definition. If Ψ and Ξ are gauge-invariant functions of field configurations ϕ_x and $A_{x,x+\hat{k}}$ on Σ, their scalar product is given by the formula:

$$(\Psi|\Xi) = \frac{1}{2\pi} \int \overline{\Psi}\Xi \prod_{(x,x+\hat{k})\notin\mathcal{T}} dA_{x,x+\hat{k}} \prod_x d\phi_x \, d\bar{\phi}_x , \tag{4.9}$$

where $\mathcal{T}$ is any three-dimensional tree in Σ.

It is easy to see that the above definition does not depend upon the specific choice of the tree $\mathcal{T}$ and the choice of a gauge on it (i.e. the choice of the values of the "along-tree gauge potentials"). In fact, we integrate over the space of gauge-orbits, parameterized by "tree-parameters". Moreover, we use the invariance of the induced measure with respect to a particular choice of the "tree-gauge".

The Hilbert space generated by the above scalar product will be denoted $\mathcal{H}_\Sigma$.

Elements of $\mathcal{O}_\Sigma$ act in a natural way as operators in $\mathcal{H}_\Sigma$. Due to constraint (4.4), the operator Q_x reduces to "two-dimensional angular momentum" operator (4.3) whose spectrum is $\{g\hbar \cdot n, n \in \mathbb{Z}\}$. Hence, the condition (4.8) has been fulfilled.

Theorem. *For a finite three-dimensional lattice Σ, there exists only one (up to unitary equivalence) representation of the algebra $\mathcal{O}_\Sigma$, which fulfills the condition (4.8).*

Proof. Let us choose any three-dimensional tree $\mathcal{T}$ in Σ. To each site x different from the root x_0 we assign the following unitary operator:

$$W_x(\mathcal{T}) := \prod_{(y,y+\hat{k})\in(x_0,x)_{\mathcal{T}}} W_{y,y+\hat{k}} . \qquad (4.10)$$

Take the following collection $\mathcal{O}_\Sigma(\mathcal{T})$ of operators:
(1) (ρ_x, K_x) at all the lattice sites,
(2) $(W_x(\mathcal{T}), (1/g)Q_x)$ at all the lattice sites different from the root x_0,
(3) $(\mathcal{F}_{x,x+\hat{k}}(\mathcal{T}), D_{x,x+\hat{k}})$ on all the off-tree links.
It is easy to see that $\mathcal{O}_\Sigma(\mathcal{T})$ may be chosen as the minimal Lie subalgebra generating the entire algebra $\mathcal{O}_\Sigma$ and that the commutation relations (4.6) are equivalent to the canonical commutation relations between pairs of operators (1)–(3):

$$\begin{aligned}
\left[\rho_x, a^3 K_x\right] &= i\hbar \mathbf{1}, \\
\left[W_x(\mathcal{T}), (1/g)Q_x\right] &= -\hbar W_x, \\
\left[(1/a)\mathcal{F}_{x,x+\hat{k}}(\mathcal{T}), a^3 D_{x,x+\hat{k}}\right] &= -i\hbar \mathbf{1},
\end{aligned} \qquad (4.11)$$

with all the remaining commutators vanishing. All the representations of such a Heisenberg algebra are equivalent to the Schrödinger representation in terms of the wave functions "$\tilde{\Psi}$" depending on the parameters $(\rho_x, W_x(\mathcal{T}), (1/a)\mathcal{F}_{x,x+\hat{k}}(\mathcal{T}))$ (the so-called θ-representations of Q_x are excluded by the condition (4.8)). The wave functions are square-integrable with respect to the Lebesgue measure on the first and the last group of parameters and with respect to the standard measure $(W_x)^{-1} dW_x$ on the circle. The momenta $(a^3 K_x, (1/g)Q_x, a^3 D_{x,x+\hat{k}})$ are represented as differentiations with respect to the above variables.

To see that the above Schrödinger representation is equivalent to the one constructed before, it is sufficient to observe that in the tree-gauge $A \equiv 0$ on $\mathcal{T}$ we have

$$\phi_x = e^{\rho_x} W_x(\mathcal{T}) \arg(\phi_{x_0})$$

outside of the tree root and

$$A_{x,x+\hat{k}} = (1/a)\mathcal{F}_{x,x+\hat{k}}(\mathcal{T})$$

on the off-tree links. We stress that the phase $\arg(\phi_{x_0})$ of ϕ_{x_0} at the tree root is completely arbitrary and the corresponding momentum $(1/g)Q_{x_0}$ is not an independent variable: its value is equal to the value of the total electric charge contained in Σ (determined by the boundary condition which we imposed) minus the sum of the remaining charges $(1/g)Q_x$ outside of the tree root.

Hence, the formula

$$\Psi = \prod_x \mathrm{e}^{-\rho_x} \tilde{\Psi}$$

defines a square-integrable function with respect to the Lebesgue measure $d\phi\, d\bar{\phi}\, dA$ which we used in the definition (4.9), i.e. an element of our Hilbert space $\mathcal{H}_\Sigma$. The transformation is obviously unitary because the integration over $\arg(\phi_{x_0})$ in (4.9) kills the factor 2π. This ends the proof.   □

4.4. Unbounded lattice

There is a natural inclusion $\mathcal{O}_{\Sigma_1} \subset \mathcal{O}_{\Sigma_2}$ when $\Sigma_1 \subset \Sigma_2$. Hence, the algebra $\mathcal{O}_\Sigma$ for the complete, unbounded lattice $\Sigma = \mathbb{Z}^3$ can be immediately obtained as the inductive limit of the algebras corresponding to bounded Σ's. Unfortunately, the representation of the commutation relations (4.6) for the unbounded lattice cannot be unique, because the representation of the Heisenberg commutation relations (4.11) is not unique for infinitely many degrees of freedom. The theory splits, therefore, into separate, non interacting sectors. Different sectors of the theory may be constructed e.g. from different vacuum states.

The space of all the (mixed) states σ_{Σ_1} for a bounded lattice Σ_1 can be treated as a dual of $\mathcal{O}_{\Sigma_1}$. The inclusion of $\mathcal{O}_{\Sigma_1}$ into $\mathcal{O}_{\Sigma_2}$ defines in a unique way the projection

$$P_{\Sigma_1,\Sigma_2} : \sigma_{\Sigma_2} \rightarrow \sigma_{\Sigma_1}.$$

The space of quantum states σ_Σ on the unbounded lattice can thus be constructed as a projective limit of the spaces corresponding to bounded Σ's. Choosing a single state $s \in \sigma_\Sigma$ as a vacuum, enables us to reconstruct the entire sector of the theory via the standard Gelfand–Naimark–Segal procedure. The physically correct choice of the vacuum will be discussed in the sequel. We stress, however, that each Hilbert space obtained this way corresponds to a fixed value of the total electric charge and the "superselection rules" are automatically satisfied. Hence, for describing the interaction of charged particles, the non-vacuum sectors will be also important.

5. Quantum dynamics

5.1. The Feynman path integral in terms of gauge-invariants

To define the evolution of the quantum system defined above, we begin with a discrete version of formula (0.2). Both the left- and right-hand sides are functions of the fields $(\phi_z, A_{z,z+\hat{k}})$ on the surface Σ_{fin}. To compute the value of the right-hand side we have to integrate over initial and intermediate configurations, with the final configurations $(\phi_z, A_{z,z+\hat{k}})$ being fixed. But, since the integrand is constant on gauge-orbits, the integration over all intermediate configurations will always be ill-defined. Similarly as in the definition of the Hilbert space $\mathcal{H}_\Sigma$, we have to define the integral in such a way that each gauge orbit is taken only once.

Definition. The quantum evolution on the lattice is given by

$$
U_{(\Sigma_{\text{fin}}, \Sigma_{\text{init}})} \Psi_{\text{init}} = \frac{1}{N} \int e^{(i/\hbar)S} \Psi_{\text{init}}(\phi_x, A_{x,x+\hat{k}}) \prod_{t_{\text{init}} < y^0 < t_{\text{fin}}} \mathrm{d}\phi_y \, \mathrm{d}\bar{\phi}_y
$$
$$
\times \prod_{(y,y+\hat{\mu}) \notin \mathcal{T}} \mathrm{d}A_{y,y+\hat{\mu}} \prod_{x \in \Sigma_{\text{init}}} \mathrm{d}\phi_x \, \mathrm{d}\bar{\phi}_x \prod_{(x,x+\hat{k}) \notin \mathcal{T}} \mathrm{d}A_{x,x+\hat{k}} , \qquad (5.1)
$$

where $\mathcal{T}$ is any four-dimensional tree, covering the intermediate region $\{t_{\text{init}} < y^0 < t_{\text{fin}}\}$ between Σ_{init} and Σ_{fin}, together with the initial surface Σ_{init}, and the corresponding tree-gauge has been chosen in an arbitrary way. The action S corresponding to the region $\{t_{\text{init}} < y^0 < t_{\text{fin}}\}$ is given by formula (3.2).

Similarly as in the case of formula (4.9) it is easy to check that the result does not depend upon the specific choice of the tree and the choice of the gauge. Moreover, because of the gauge invariance of the initial wave function and the action S, the result is a gauge-invariant function of final parameters, i.e. it describes a physical quantum state on Σ_{fin}. Hence, we have defined an operator $U_{(\Sigma_{\text{fin}}, \Sigma_{\text{init}})}$ from $\mathcal{H}_{\Sigma_{\text{init}}}$ to $\mathcal{H}_{\Sigma_{\text{fin}}}$. We will show in the sequel, that the normalization factor N may be chosen in such a way, that U is unitary.

5.2. Factorization of the evolution

The simplest way to compute the value of the integral (5.1) consists in choosing the four-dimensional tree $\mathcal{T}$ which is composed of a three-dimensional tree $\mathcal{T}_{\text{init}}$ in Σ_{init} and all the time-like links $(y, y+\hat{0})$. Moreover, we choose a zero temporal gauge (for the moment we do not need to specify the value of the gauge on $\mathcal{T}_{\text{init}}$). First, let us perform the integration over the intermediate configurations. This way we obtain a kernel depending upon the initial and the final configurations. But the action S is a sum of kinetic terms

$$
S_{\text{kin}} = \frac{a^3}{2\tau} \sum_y |\phi_{y+\hat{0}} - \phi_y|^2 + \frac{a^3}{2\tau} \sum_{y,k} (A_{y+\hat{0}, y+\hat{0}+\hat{k}} - A_{y, y+\hat{k}})^2 ,
$$

assigned to each elementary time-interval of the discretized time axis, and potential terms

$$S_{\text{pot}} = -\tau a^3 \sum_y V(|\phi_y|^2) - \frac{1}{2}\tau a^3 \sum_{y,k} |(D\phi)_{y,y+\hat{k}}|^2 - \frac{1}{4}\tau a^3 \sum_{y,k,l} (B_{y;\hat{k},\hat{l}})^2 \ ,$$

assigned to each surface $\{y^0 = \text{const.}\}$. Hence, the kernel $U_{(\Sigma_{\text{fin}}, \Sigma_{\text{init}})}$ is equal to the successive superposition of the kinetic kernels $U_{\text{kin}} = e^{(i/\hbar)S_{\text{kin}}}/N$ and the potential kernels $U_{\text{pot}} = e^{(i/\hbar)S_{\text{pot}}}$.

Observe that the kinetic kernel is equal to the resolvent kernel corresponding to the linear Schrödinger equation:

$$U_{(t+\tau, t)}(y, x) = \frac{1}{\sqrt{2\pi\hbar\tau}} e^{(i/\hbar\tau)(x-y)^2} \ .$$

The kinetic evolution consists, therefore, in the free Schrödinger evolution during the time interval τ of *all the degrees of freedom* $(\phi_y, A_{y,y+\hat{k}})$ on Σ. Such an evolution kernel is obviously unitary when applied to any square-integrable function of initial configurations $(\phi_x, A_{x,x+\hat{k}})$. Applying this kernel means integrating *over all* the parameters $(\phi_x, A_{x,x+\hat{k}})$. The normalization factor equals $(2\pi\hbar\tau)^{-1/2}$ for each degree of freedom.

In our case we are going to apply the kernel to gauge-invariant wave functions only. This means that, according to formula (5.1), we are going to integrate over the off-tree parameters only. We stress, however, that this difference does not produce any problem, because both integrals give the same value when applied to a gauge-invariant wave function. This is due to the oscillatory (Fresnel-like) character of the kernel. The integration of U_{kin} with respect to the gauge parameters over each gauge orbit will give us the value 1, so only the integration over the space of orbits remains. The situation is similar to the following example: we may apply the three-dimensional free Schrödinger evolution to wave functions which do not depend upon the z-variable. The integration over dz does not produce any difficulty due to the oscillatory character of the Fresnel kernel.

The potential evolution, consisting in multiplying the wave function by U_{pot}, is obviously unitary because $|U_{\text{pot}}| = 1$.

5.3. Quantum evolution in the Heisenberg picture

It is very instructive to reconsider the above evolution in the Heisenberg picture, where the quantum state remains constant in time and the quantum observables evolve according to the formula $\mathcal{O} \rightarrow \tilde{\mathcal{O}} := U^{-1}\mathcal{O}U$. It is obvious that the potential evolution leaves the configurations (R, W, B) invariant. The evolution of momenta (K, D) may be easily obtained, according to formulae:

$$K^+ \Psi = U_{\text{pot}}^{-1} K^- U_{\text{pot}} \Psi \ , \qquad D^+ \Psi = U_{\text{pot}}^{-1} D^- U_{\text{pot}} \Psi \ .$$

This way we obtain the following formulae, analogous to classical equations (3.15):

$$R^+_x = R^-_x \, ,$$

$$W^+_{x,x+\hat{k}} = W^-_{x,x+\hat{k}} \, ,$$

$$B^+_{x;\hat{k},\hat{l}} = B^-_{x;\hat{k},\hat{l}} \, , \tag{5.2}$$

$$K^+_x = K^-_x - 2\tau V'\left(R^2_x\right) \cdot R^2_x - \frac{\tau}{a^2} \sum_{\hat{k}} \left(R^2_x - R_x R_{x+\hat{k}} \frac{W_{x,x+\hat{k}} + W^{-1}_{x,x+\hat{k}}}{2}\right) ,$$

$$D^+_{x,x+\hat{k}} = D^-_{x,x+\hat{k}} + g\frac{\tau}{a} R_x R_{x+\hat{k}} \frac{W_{x,x+\hat{k}} - W^{-1}_{x,x+\hat{k}}}{2i} + \frac{\tau}{a} \sum_{\hat{l}} B_{x;\hat{k},\hat{l}} \, .$$

In a similar way we may obtain the potential evolution of the electric charge M given by (4.3):

$$M^+_x = U^{-1}_{\text{pot}} M^-_x U_{\text{pot}} = M^-_x + \frac{i}{\hbar}(M^-_x S_{\text{pot}})$$

$$= M^-_x + \frac{\tau}{a^2} R_x \sum_{\hat{k}} R_{x+\hat{k}} \frac{W_{x,x+\hat{k}} - W^{-1}_{x,x+\hat{k}}}{2i} \, .$$

This equation is the lattice version of the continuity equation: the value of the electric charge M changes only by the divergence of the electric current carried by the matter field.

The corresponding formulae for the kinetic evolution will be derived in Section 6. Here we give only the final results. To simplify our notation we introduce the following quantity, analogous to the classical $W_{x,x+\hat{0}}$, defined in (3.22):

$$W_{x,x+\hat{0}} = W^{-1}_{x+\hat{0},x} = \left(\phi_x R^{-1}_x\right)^{-1} \phi_{x+\hat{0}} R^{-1}_{x+\hat{0}}$$

$$= R_x \left[1 + \tau R^{-1}_x \left(K^+_x + iM^+_x\right) R^{-1}_x\right] \left(R^2_x + 2\tau K^+_x + 2\tau^2 H^+_x\right)^{-1/2} . \tag{5.3}$$

It will be proved in Section 6 that the same quantity may be expressed in terms of the Cauchy data on Σ^-:

$$W_{x,x+\hat{0}} = \left(R^2_{x+\hat{0}} - 2\tau K^-_{x+\hat{0}} + 2\tau^2 H^-_{x+\hat{0}}\right)^{-1/2}$$

$$\times \left[1 - \tau R^{-1}_{x+\hat{0}} \left(K^-_{x+\hat{0}} - iM^-_{x+\hat{0}}\right) R^{-1}_{x+\hat{0}}\right] R_{x+\hat{0}} \, .$$

Furthermore, we denote as usual:

$$M_x = \frac{1}{ga} \sum_{\hat{k}} D_{x,x+\hat{k}} \quad \text{and} \quad H_x = \tfrac{1}{2} R^{-1}_x \left(K^2_x + M^2_x\right) R^{-1}_x \, ,$$

on both Σ^+ and Σ^-.

The kinetic evolution now reads

$$R_{x+\hat{0}} = \left(R_x^2 + 2\tau K_x^+ + 2\tau^2 H_x^+ \right)^{1/2} ,$$

$$W_{x+\hat{0},x+\hat{0}+\hat{k}} = W_{x+\hat{0},x} \exp \left(-\mathrm{i} g a \frac{\tau}{2} D_{x,x+\hat{k}}^+ \right) W_{x,x+\hat{k}}$$

$$\times \exp \left(-\mathrm{i} g a \frac{\tau}{2} D_{x,x+\hat{k}}^+ \right) W_{x+\hat{k},x+\hat{0}+\hat{k}} , \qquad (5.4)$$

$$B_{x+\hat{0};\hat{k},\hat{l}} = B_{x;\hat{k},\hat{l}} - \frac{\tau}{a} \left(D_{x,x+\hat{k}}^+ + D_{x+\hat{k},x+\hat{k}+\hat{l}}^+ + D_{x+\hat{k}+\hat{l},x+\hat{l}}^+ + D_{x+\hat{l},x}^+ \right) ,$$

$$K_{x+\hat{0}}^- = K_x^+ + 2\tau H_x^+ ,$$

$$D_{x+\hat{0},x+\hat{0}+\hat{k}}^- = D_{x,x+\hat{k}}^+ .$$

We stress that the quantum evolution is given by the same equations as in the classical case (formulae (3.19)–(3.23)). However, the evolution for W is given by the product of non-commuting operators. Changing their order changes the result completely. In (5.4) we have chosen an ordering, which formally reproduces the classical equation (3.23).

Field equations (5.2) and (5.4) contain the complete set of Maxwell equations. In fact, div $B = 0$ is an identity. The kinetic evolution reproduces the equation $\dot{B} = -\mathrm{curl}\, D$. The potential evolution reproduces $\dot{D} = -j + \mathrm{curl}\, B$, and the last equation div $D = q$ is the definition of the electric charge density.

We stress that, similarly as in the classical version of the theory, the quantum dynamics is causal. Indeed, the evolution of a momentum at the site x depends on variables at the nearest neighbors of x only. This way, knowing the quantum observables in a bounded region of Σ, we can compute the time-evolved operators in an entire future causal shadow of this region.

5.4. Self-consistency of the evolution

In the evolution formulae (5.2) and (5.4) we have applied a potentially dangerous operation, the square root, to a time-dependent operator. To prove that our formulae are formally correct, we have to show that this operator is non-negative for all times.

Theorem. *The operator $R^2 + 2\tau K + 2\tau^2 H$ is a positive, self-adjoint operator for all values of τ.*

Proof. The operator in question is equal to the square of the modulus of the following operator: $N = R + \tau R^{-1}(K + \mathrm{i}M + \mathrm{i}\hbar)$. Indeed, we have

$$N^\dagger N = \left[R + \tau(K - \mathrm{i}M - \mathrm{i}\hbar)R^{-1} \right]\left[R + \tau R^{-1}(K + \mathrm{i}M + \mathrm{i}\hbar) \right]$$

$$= R^2 + 2\tau K + \tau^2 (K - \mathrm{i}M - \mathrm{i}\hbar)R^{-2}(K + \mathrm{i}M + \mathrm{i}\hbar)$$

$$= R^2 + 2\tau K + 2\tau^2 H . \qquad \square$$

We also have the following theorem.

Theorem. *The operator $\tilde{W}$, given by the right-hand side of (5.4) remains unitary if W was unitary.*

Proof. Unitarity of the evolved $W_{x,x+\hat{k}}$ follows from formula (5.4) if we prove unitarity of $W_{x,x+\hat{0}}$,

$$
\begin{aligned}
W^{\dagger}_{x,x+\hat{0}} W_{x,x+\hat{0}} &= \left(R_x^2 + 2\tau K_x^+ + 2\tau^2 H_x^+ \right)^{-1/2} \left[R_x + \tau R_x^{-1} \left(K_x^+ - iM_x^+ \right) \right] \\
&\quad \times \left[R_x + \tau \left(K_x^+ + iM_x^+ \right) R_x^{-1} \right] \left(R_x^2 + 2\tau K_x^+ + 2\tau^2 H_x^+ \right)^{-1/2} \\
&= \left(R_x^2 + 2\tau K_x^+ + 2\tau^2 H_x^+ \right)^{-1/2} \left[R_x^2 + 2\tau K_x^+ \right. \\
&\quad \left. + \tau^2 R_x^{-1} \left((K_x^+)^2 + (M_x^+)^2 \right) R_x^{-1} \right] \left(R_x^2 + 2\tau K_x^+ + 2\tau^2 H_x^+ \right)^{-1/2} \\
&= 1 . \qquad\qquad \square
\end{aligned}
$$

This proves that Eqs. (5.2) and (5.4) define a self-consistent quantum dynamics of our algebra of observables.

5.5. Boundary conditions

Quantum evolution of Heisenberg operators described by formulae (5.2)–(5.4) is causal. This statement means that we may solve formally the initial value problem for quantum operators in the same way as we did for the classical gauge-invariants. This way, each local quantum observable may be represented as a combination of the observables contained in its discrete "past light cone". In the above statement the "speed of light" is understood in a purely combinatorial sense, corresponding to the structure of dynamical equations (5.2)–(5.4): one lattice spacing in the direction of time corresponds to at most one lattice spacing in the direction of space.

The above procedure cannot, however, be continued if the above "light cone" hits the boundary of our finite lattice Σ. Similarly as in classical case, to continue the solution, one has to impose appropriate boundary conditions. These conditions are of purely classical ("c-number") character. They will be imposed on a three-dimensional surface $\mathcal{B}$ defined as Cartesian product $T \times \partial \Sigma$ of the time axis T and the two-dimensional boundary of Σ. The observables assigned to elements of the lattice $\mathcal{B}$ are thus non-dynamical and purely classical.

The boundary data which we have to fix on $\mathcal{B}$ are R_x and $W_{x,x+\hat{0}}$ at all the nodes $x \in \mathcal{B}$ and $D_{x,x+\hat{k}}$ at all the links $(x, x + \hat{k})$ such that both x and $x + \hat{k}$ belong to $\mathcal{B}$.

To prove that the dynamics is now well defined we first analyze the potential evolution. The configurations R, W and B do not change during the potential evolution (5.2). To determine the momentum K_x^+ we need K_x^-, $R_{x+\hat{k}}$ in the space-like neighbors of x and $W_{x,x+\hat{k}}$ on all the space-like links starting at x. The boundary value of R_y being fixed, the evolution of K is, therefore, well defined. The potential evolution of the electric induction D is determined by the "internal" (i.e. dynamical) data.

The kinetic evolution of internal momenta K_x and $D_{x,x+\hat{k}}$ uses only internal quantities. The same holds for R_x. To determine $W_{x+\hat{0},x+\hat{0}+\hat{k}}$ we need $D_{x,x+\hat{k}}$ as well as both $W_{x,x+\hat{0}}$ and $W_{x+\hat{k},x+\hat{0}+\hat{k}}$. At the internal lattice nodes we may compute $W_{x,x+\hat{0}}$ in terms of R_x, K_x and $M_x = (1/ga) \sum_{\hat{k}} D_{x,x+\hat{k}}$ but we have to fix its value at the boundary. Finally, to be able to determine the magnetic field B on a future internal plaquette we need to know the electric induction on all the boundary of the respective plaquette. Hence, we have to use the value D inside the boundary $\mathcal{B}$.

This ends the consistency proof of the field dynamics.

6. Kinetic evolution of observables

6.1. Kinetic evolution of matter fields

To calculate the kinetic evolution of our observables, we take first the free Schrödinger evolution of all the degrees of freedom (ψ, A, π, D) in the temporal gauge $A_{x,x+\hat{0}} = 0$. Finally, we will use these results to compute the evolution of their gauge-invariant combinations (R, W, B, K, D). This way we obtain results, which are already gauge independent.

We stress that the "free evolution of ϕ and A", which we use in this section, is only a convenient way of calculation and has *no physical significance*. In fact, in our approach *there is no way* to give any reasonable meaning to quantities ϕ and A on the quantum level.

The free Schrödinger evolution of the two degrees of freedom (Re ϕ_x, Im ϕ_x) carried by ϕ_x may be rewritten in terms of the operators (R, F, K, M), where $\phi = F \cdot R$ is the polar decomposition of the normal operator $\phi = \phi_1 + i\phi_2$. These operators fulfill the following commutation relations:

$$[R, K] = i\hbar a^{-3} R , \qquad [F, M] = -\hbar a^{-3} F ,$$

all other commutators are equal to zero.

The free Hamiltonian (equal to the two-dimensional Laplacian in the variable ϕ) may be written as follows:

$$H = -\tfrac{1}{2}\hbar^2 \Delta_\phi = \tfrac{1}{2}(\pi_1^2 + \pi_2^2) = \tfrac{1}{2}R^{-1}(K^2 + M^2)R^{-1} ,$$

$a^3 H$ being the generator of the time evolution. The evolution of the field operators in the Heisenberg picture is given by the equations

$$\dot{R} = \frac{i}{\hbar}[a^3 H, R] = R^{-1}\Big(K + i\frac{\hbar}{2a^3}\Big), \qquad \dot{F} = \frac{i}{\hbar}[a^3 H, F] = iR^{-2}\Big(M - \frac{\hbar}{2a^3}\Big)F,$$

$$\dot{K} = \frac{i}{\hbar}[a^3 H, K] = 2H, \qquad \dot{M} = \frac{i}{\hbar}[a^3 H, M] = 0 .$$

The Hamiltonian H and the angular momentum M are constant in time. Hence, the evolution of K is linear:

$$K(t) = K(0) + 2H(0)t .$$

Moreover, for $C = R^2$ we have

$$\dot{C} = \frac{i}{\hbar}[a^3 H, C] = 2K ,$$

which implies

$$C(t) = C(0) + 2K(0)t + 2H(0)t^2 .$$

We already proved that the above operator remains always positive. To find $R(t)$ we have to take its square root.

To find the evolution of F let us consider the product RF. We have

$$(RF)^{\cdot} = \frac{i}{\hbar}[a^3 H, RF] = R^{-1}(K + iM)F .$$

Observe that the right-hand side commutes with the Hamiltonian and, therefore, is constant in time. Hence, we have

$$R(t)F(t) = \left[R(0) + R(0)^{-1} (K(0) + iM(0)) t \right] F(0) .$$

Summarizing, after the time τ we have

$$H(\tau) = H(0), \qquad K(\tau) = K(0) + 2\tau H(0),$$
$$R(\tau) = \left(R(0)^2 + 2\tau K(0) + 2\tau^2 H(0) \right)^{1/2}, \qquad M(\tau) = M(0),$$
$$F(\tau) = \left(R(0)^2 + 2\tau K(0) + 2\tau^2 H(0) \right)^{-1/2}$$
$$\times \left[R(0) + \tau R(0)^{-1} (K(0) + iM(0)) \right] F(0).$$

6.2. Kinetic evolution of gauge fields

In our temporal gauge each degree of freedom $A_{x,x+\hat{k}}$ undergoes separately the one-dimensional, free Schrödinger evolution and its canonical momentum $D = -(\hbar/ia^3)(\partial/\partial A)$ remains constant. Hence, after time τ we have

$$A(\tau) = A(0) + \frac{\hbar\tau}{ia^3} \frac{\partial}{\partial A} = A(0) - \tau D(0) .$$

Taking the lattice curl of both sides we have

$$B(\tau) = B(0) - \tau \operatorname{curl} D(0) ,$$

reproducing the free Maxwell equation.

6.3. Kinetic evolution of physical observables

Using the evolution of gauge-dependent fields, we may finally find the evolution of the gauge-invariant quantity

$$W_{x,x+\hat{k}}(t) = \exp(iga A_{x,x+\hat{k}}(t)) F_{x+\hat{k}}(t) F_x^{-1}(t)$$

(no ordering problem because the operators on the right-hand side commute). Moreover, according to (5.3), we have

$$W_{x,x+\hat{0}} = \left(\phi_x R_x^{-1}\right)^{-1} \phi_{x+\hat{0}} R_{x+\hat{0}}^{-1} = F_x^{-1} F_{x+\hat{0}} \,.$$

Using the previous results it is easy to compute

$$\begin{aligned}
W_{x+\hat{0},x+\hat{0}+\hat{k}} &= F_{x+\hat{0}}^{-1} \exp(iga A_{x+\hat{0},x+\hat{0}+\hat{k}}) \, F_{x+\hat{0}+\hat{k}} \\
&= \left(F_{x+\hat{0}}^{-1} F_x\right) F_x^{-1} \exp\left(-iga\frac{\tau}{2} D_{x,x+\hat{k}}^+\right) \exp(iga A_{x,x+\hat{k}}) \\
&\quad \times \exp\left(-iga\frac{\tau}{2} D_{x,x+\hat{k}}^+\right) F_{x+\hat{k}} \left(F_{x+\hat{k}}^{-1} F_{x+\hat{0}+\hat{k}}\right) \\
&= W_{x,x+\hat{0}}^{-1} \exp\left(-iga\frac{\tau}{2} D_{x,x+\hat{k}}^+\right) W_{x,x+\hat{k}} \\
&\quad \times \exp\left(-iga\frac{\tau}{2} D_{x,x+\hat{k}}^+\right) W_{x+\hat{k},x+\hat{0}+\hat{k}} \,,
\end{aligned}$$

which immediately implies (5.4).

7. Formulation in terms of configuration variables

7.1. Algebra of observables

The entire theory can also be rewritten in terms of field configurations only, without any use of field momenta. To keep a complete set of initial data, we have to replace the information which was contained in the momenta by the information carried by the field configuration on an adjacent Σ. This way, we can describe the observable algebra by the field configurations over two consecutive space-like hyperplanes Σ_t and $\Sigma_{t+\hat{0}}$, together with all the time-like links and plaquettes between them. In this formulation it is convenient to use the following generators of the algebra:

$$r_x = R_x^2 \quad \text{and} \quad w_{x,x+\hat{\mu}} = R_x W_{x,x+\hat{\mu}} R_{x+\hat{\mu}} \,,$$

which, in terms of gauge-dependent fields, reads:

$$w_{x,x+\hat{\mu}} = \overline{\phi_x} \exp(iga A_{x,x+\hat{\mu}}) \phi_{x+\hat{\mu}} \,.$$

Of course,

$$w_{x+\hat{\mu},x} = \overline{w_{x,x+\hat{\mu}}} \,.$$

It is easy to obtain the following, complete set of commutation relations between these generators:

$$\left[r_x, r_{x+\hat{0}}\right] = \frac{2i\hbar\tau}{a^3} \left(w_{x,x+\hat{0}} + \overline{w_{x,x+\hat{0}}}\right),$$

$$\left[r_x, w_{x,x+\hat{0}}\right] = \left[r_x, \overline{w_{x,x+\hat{0}}}\right] = \frac{2i\hbar\tau}{a^3} r_x,$$

$$\left[r_{x+\hat{0}}, w_{x,x+\hat{0}} \right] = \left[r_{x+\hat{0}}, \overline{w_{x,x+\hat{0}}} \right] = -\frac{2i\hbar\tau}{a^3} r_{x+\hat{0}},$$

$$\left[r_{x+\hat{0}}, w_{x,x+\hat{k}} \right] = -\frac{2i\hbar\tau}{a^3} \overline{w_{x,x+\hat{0}}} \, r_x^{-1} \, w_{x,x+\hat{k}},$$

$$\left[r_x, w_{x+\hat{0},x+\hat{0}+\hat{k}} \right] = \frac{2i\hbar\tau}{a^3} w_{x,x+\hat{0}} \, r_{x+\hat{0}}^{-1} \, w_{x+\hat{0},x+\hat{0}+\hat{k}},$$

$$\left[w_{x,x+\hat{0}}, w_{x,x+\hat{k}} \right] = -\frac{2i\hbar\tau}{a^3} w_{x,x+\hat{k}},$$

$$\left[\overline{w_{x,x+\hat{0}}}, w_{x,x+\hat{k}} \right] = 0,$$

$$\left[w_{x,x+\hat{0}}, w_{x+\hat{0},x+\hat{0}+\hat{k}} \right] = 0,$$

$$\left[\overline{w_{x,x+\hat{0}}}, w_{x+\hat{0},x+\hat{0}+\hat{k}} \right] = -\frac{2i\hbar\tau}{a^3} w_{x+\hat{0},x+\hat{0}+\hat{k}},$$

$$\left[w_{x+\hat{k},x+\hat{k}+\hat{l}}, w_{x+\hat{0},x+\hat{0}+\hat{k}} \right] = \frac{2i\hbar\tau}{a^3} w_{x+\hat{k},x+\hat{k}+\hat{l}} \, w_{x+\hat{0},x+\hat{0}+\hat{k}}$$
$$\times r_{x+\hat{0}+\hat{k}}^{-1} \, \overline{w_{x+\hat{k},x+\hat{0}+\hat{k}}} \, r_{x+\hat{k}}^{-1},$$

$$w_{x+\hat{0},x+\hat{0}+\hat{k}} \, w_{x,x+\hat{k}} = \exp\left(i\hbar g^2 \frac{\tau}{a} \right) w_{x,x+\hat{k}} \, w_{x+\hat{0},x+\hat{0}+\hat{k}},$$

$$w_{x+\hat{0},x+\hat{0}+\hat{k}} \, w_{x+\hat{k},x} = \exp\left(i\hbar g^2 \frac{\tau}{a} \right) w_{x+\hat{k},x} \left(\overline{w_{x,x+\hat{0}}} - \frac{2i\hbar\tau}{a^3} \right) r_x^{-1} \, w_{x,x+\hat{0}}$$
$$\times r_{x+\hat{0}}^{-1} w_{x+\hat{0},x+\hat{0}+\hat{k}} \, r_{x+\hat{0}+\hat{k}}^{-1} \, \overline{w_{x+\hat{k},x+\hat{0}+\hat{k}}}$$
$$\times r_{x+\hat{k}}^{-1} \left(w_{x+\hat{k},x+\hat{0}+\hat{k}} - \frac{2i\hbar\tau}{a^3} \right)$$

7.2. Second-order field equations

Field equations may also be rewritten as second-order difference equations for the independent set of configuration variables r_x and $w_{x,x+\hat{\mu}}$ described above. At every lattice site we will have an equation analogous to the Klein–Gordon-like equation (1.7) for the "matter" field r and at every lattice link an equation analogous to the Maxwell equation (1.7) for the "electromagnetic" field.

First notice that the fourth formula of Eqs. (5.2) is already in a form very similar to the classical expression. We will express the momentum K in terms of the configuration variables using the definition (5.3) of $W_{x,x+\hat{0}}$. Indeed, we have

$$w_{x,x+\hat{0}} = R_x^2 + \tau \left(K_x^+ + iM_x^+ + i\frac{\hbar}{a^3} \right).$$

Taking the real and the imaginary parts we get

$$K_x^+ = \frac{1}{\tau} \left[\frac{w_{x,x+\hat{0}} + \overline{w_{x,x+\hat{0}}}}{2} - r_x \right] \tag{7.1}$$

and

$$M_x^+ = \frac{1}{\tau} \frac{w_{x,x+\hat{0}} - \overline{w_{x,x+\hat{0}}}}{2i} - \frac{\hbar}{a^3} \, . \tag{7.2}$$

In the same way we obtain

$$K_x^- = -\frac{1}{\tau} \left[\frac{w_{x,x-\hat{0}} + \overline{w_{x,x-\hat{0}}}}{2} - r_x \right] \quad \text{and} \quad M_x^- = -\frac{1}{\tau} \frac{w_{x,x-\hat{0}} - \overline{w_{x,x-\hat{0}}}}{2i} - \frac{\hbar}{a^3} \, .$$

Substituting the above expressions for K to (5.2), we have

$$-V'(r_x) \cdot r_x^{1/2} + \sum_{\hat{\mu}} \frac{1}{a(\mu)^2} \left[\frac{w_{x,x+\hat{\mu}} + \overline{w_{x,x+\hat{\mu}}}}{2} - r_x \right] = 0 \, , \tag{7.3}$$

which is the lattice version of the first equation of (1.7) for the quantum observables.

To obtain the analog of the second of Eqs. (1.7), let us choose a space-like lattice link $(x, x + \hat{k})$. The last equation of (5.2) is completely analogous to the classical expression, so we merely rewrite it in another form:

$$\frac{1}{a} \sum_{\hat{l}} B_{x;\hat{k},\hat{l}} - \frac{1}{\tau} D_{x,x+\hat{k}}^+ + \frac{1}{\tau} D_{x,x+\hat{k}}^- = -g \frac{1}{a} \frac{w_{x,x+\hat{k}} - \overline{w_{x,x+\hat{k}}}}{2i} \, . \tag{7.4}$$

To do the same for time-like links, we substitute (7.2) to the definition (4.7) of M:

$$\frac{1}{a} \sum_{\hat{k}} D_{x,x+\hat{k}}^+ = g \frac{1}{\tau} \frac{w_{x,x+\hat{0}} - \overline{w_{x,x+\hat{0}}}}{2i} - g \frac{\hbar}{a^3}$$

$$= g \frac{1}{\tau} r_x^{-1/2} \frac{w_{x,x+\hat{0}} - \overline{w_{x,x+\hat{0}}}}{2i} r_x^{1/2} \, . \tag{7.5}$$

Summarizing, we may write the expressions (7.4) and (7.5) in one formula:

$$\sum_{\hat{v}} \frac{g^{vv}}{a(v)} f_{x;\hat{\mu},\hat{v}} = -g \frac{1}{a(\mu)} r_x^{-1/2} \frac{w_{x,x+\hat{\mu}} - \overline{w_{x,x+\hat{\mu}}}}{2i} r_x^{1/2} \, ,$$

analogous to the second equation of (1.7).

Finally, the value of the field $f_{x;\hat{\mu};\hat{v}}$ may be expressed in terms of r and w (up to $2\pi n / ga(\mu)a(v)$). For both μ and v being space-like, this is given simply by the constraint (2.9). If one of them is time-like, this is given by the second of Eqs. (5.4):

$$\exp \left(i\hbar g^2 \frac{\tau}{2a} \right) w_{x+\hat{k},x} r_x^{-1} w_{x,x+\hat{0}} r_{x+\hat{0}}^{-1}$$

$$\times w_{x+\hat{0},x+\hat{0}+\hat{k}} r_{x+\hat{0}+\hat{k}}^{-1} w_{x+\hat{0}+\hat{k},x+\hat{k}} r_{x+\hat{k}}^{-1} = \exp \left(-iga\tau D_{x,x+\hat{k}}^+ \right) \, ,$$

or, equivalently

$$\exp \left(-i\hbar g^2 \frac{\tau}{2a} \right) w_{x,x+\hat{0}} r_{x+\hat{0}}^{-1} w_{x+\hat{0},x+\hat{0}+\hat{k}} r_{x+\hat{0}+\hat{k}}^{-1}$$

$$\times w_{x+\hat{0}+\hat{k},x+\hat{k}} r_{x+\hat{k}}^{-1} w_{x+\hat{k},x} r_x^{-1} = \exp \left(-iga\tau D_{x,x+\hat{k}}^+ \right) \, .$$

The above equations are equivalent to the definition of $f_{\mu v}$ as the curl of v_μ, up to a vortex carried by the gradient of $\arg \phi$ (see Eq. (1.6)).

8. Conclusions and perspectives

Each step of the kinetic evolution is given by the unitary operator $e^{i\tau H_{kin}}$, where

$$H_{kin} := \frac{a^3}{2} \sum_x R_x^{-1} \left[K_x^2 + \left(\frac{1}{ga^3} Q_x \right)^2 \right] R_x^{-1} + \frac{a^3}{2} \sum_{x,k} D_{x,x+\hat{k}}^2 \cdot$$

Similarly, each step of the potential evolution is given by the unitary operator $e^{i\tau H_{pot}}$, where

$$H_{pot} := a^3 \sum_x V \left(R_x^2 \right)$$

$$+ \frac{a}{2} \sum_{x,k} \left(R_x^2 + R_{x+\hat{k}}^2 - 2 R_x R_{x+\hat{k}} \frac{W_{x,x+\hat{k}} + W_{x,x+\hat{k}}^{-1}}{2} \right) + \frac{a^3}{2} \sum_{x;k,l} B_{x;\hat{k},\hat{l}}^2 \cdot$$

Passing to the limit $\tau \to 0$ we obtain a finite-dimensional quantum mechanical system with the degrees of freedom described by the algebra $\mathcal{O}_\Sigma$. Due to the Trotter formula, the dynamics of the system is defined by the total Hamiltonian

$$H = H_{kin} + H_{pot} \cdot$$

The Hamiltonian is manifestly positive. For a finite lattice one should first find its lowest-energy state, which plays the role of the physical vacuum. Because of the high non-linearity of the theory, only numerical analysis will probably be possible. The physical vacuum has nothing to do with the *perturbative vacuum*, which may be defined as a tensor product of the free-electromagnetic-vacuum and the free-matter-vacuum. For the sake of convenience one can subtract the vacuum energy from the Hamiltonian.

Then, one has to renormalize the mass and possibly other physical parameters in the original potential V, i.e. to relate them with their physical values. To calculate the physical mass of the particles in our theory we cannot use the one-particle states as in the perturbative approach. Indeed, there is no "creation operator" in our theory, because of the super-selection rules, which are automatically fulfilled. Instead, the operator $W_{x,x+\hat{k}}$ creates a particle carrying the charge $-e$ at x and an antiparticle carrying the charge $+e$ at $x + \hat{k}$. Therefore, one has to proceed as follows. Fix a pair of sites $(x, y) \in \Sigma$. In the subspaces of all quantum states, such that $Q_x = +e$, $Q_y = -e$, and $Q = 0$ elsewhere, we find the state of the lowest energy $E(x, y)$. The maximal value of the function $E(x, y)$ with respect to x and y will be identified with twice the physical mass of the particles we want to describe. This value has to be fitted by the parameters of V.

Having chosen appropriate parameters of the theory, one may further investigate the properties of the Hamiltonian, possibly by numerical methods. In particular, dynamical problems may be solved discretizing again the time and using the unitary evolution developed in this paper.

Finally, one has to check, whether or not the physical properties of the above system depend considerably on the volume and the spacing a of the lattice used.

A similar program for Quantum Electrodynamics with spinorial matter is now under investigation and will be presented in another paper.

Acknowledgements

References

[1] A. Ashtekar and R.S. Tate, An algebraic extension of Dirac quantization: Examples, Preprint CGPG-94/6-1, gr-qc/9405073 (1994).

[2] O. Babelon and C.M. Viallet, The geometrical interpretation of the Faddeev–Popov determinant, Phys. Lett. B 85 (1979) 246.

[3] I. Białynicki-Birula and Z. Białynicka-Birula, Magnetic monopoles in the hydrodynamic formulation of quantum mechanics, Phys. Rev. D 3 (1971) 2410.

[4] I. Białynicki-Birula and P. Kurzepa, Comment on quantization in the temporal gauge, Phys. Rev. D 29 (1984) 3000.

[5] P. Chernoff and J. Marsden, Properties of Infinite Dimensional Hamiltonian Systems, Lecture Notes in Math., Vol. 425 (Springer, Berlin, 1974).

[6] P.A.M. Dirac, Quantized singularities in the electromagnetic field, Proc. Roy. Soc. A 133 (1931) 60.

[7] R. Haag, Local Quantum Physics. Fields, Particles, Algebras (Springer, Berlin, 1991).

[8] J. Kijowski and G. Rudolph, Gitterapproximation der skalaren Elektrodynamik, Wiss. Z. Karl-Marx-Univ, Leipzig, Math.- Naturwiss. R. 30(6) (1981) 613.

[9] J. Kijowski and G. Rudolph, Canonical structure of the theory of gauge fields interacting with matter fields, Rep. Math. Phys. 20 (1984) 385.

[10] J. Kijowski and G. Rudolph, New lattice approximation of gauge theories, Phys. Rev. D 31 (1985) 856.

[11] J. Kijowski and G. Rudolph, Hydrodynamical description of theories of gauge fields interacting with matter fields, Rep. Math. Phys. 21(3) (1985) 309.

[12] J. Kijowski and G. Rudolph, A non-abelian Higgs model on the lattice in terms of a complete system of gauge invariants, Lett. Math. Phys. 15 (1988) 119.

[13] J. Kijowski and G. Rudolph, The functional integral on the gauge orbit space for a non-Abelian Higgs model, Nucl. Phys. B 325 (1989) 211.

[14] J. Kijowski and G. Rudolph, Spinor electrodynamics in terms of gauge invariant quantities, Lett. Math. Phys. 29 (1993) 103.

[15] J. Kijowski and G. Rudolph, Functional integral of QED in terms of gauge invariant quantities, Lett. Math. Phys. 29 (1995) 139.

[16] J. Kijowski and A. Thielmann, On a formulation of Feynman integrals over gauge orbit space, in preparation.

[17] K. Kuchař, Hamiltonian dynamics of gauge systems, Phys. Rev. D 34(10) (1986) 3031; Covariant factor ordering of gauge systems, Phys. Rev. D 34(10) (1986) 3044.

[18] K.G. Wilson, Confinement of quarks, Phys. Rev. D 10(8) (1974) 2455.

[19] S. Zakrzewski, On the lattice approximation and Feynman path integrals for gauge fields, Thesis, Warsaw (1980), unpublished.

[20] S. Zakrzewski, Unitary relations, Publicações de Física Matemática 4 (1985), Coimbra.